

Characterizing minimum-length coordinated motions for two discs

David Kirkpatrick*

Paul Liu†

Abstract

We study the problem of planning coordinated motions for two disc robots in an otherwise obstacle-free plane. We give a characterization of collision-avoiding motions that minimize the total trace length of the disc centres, for all initial and final configurations of the robots. The individual traces are composed of at most six (straight or circular-arc) segments, and their total length can be expressed as a simple integral with a closed form solution depending only on the initial and final configuration of the robots.

1 Introduction

We consider the problem of planning collision-free motions for two disc robots of arbitrary radius in an otherwise obstacle-free environment. Given two discs \mathbb{A} and \mathbb{B} in the plane, with specified initial and final configurations, we seek a shortest collision-free motion taking \mathbb{A} and \mathbb{B} from their initial to their final configurations. The length of such a motion is defined to be the length sum of paths traced by the centres of \mathbb{A} and \mathbb{B} .

The consideration of disc robots in motion planning has amassed a substantial body of research, the bulk of which is focused on the feasibility, rather than optimality, of motions. Schwartz and Sharir [4] were the first to study motion planning for k discs among n polygonal obstacles. For $k = 2$, they developed an $\mathcal{O}(n^3)$ algorithm (later improved to $\mathcal{O}(n^2)$ [5, 8]) to determine if a collision-free motion connecting two specified configurations is feasible. When the number of robots k is unbounded, Spirakis and Yap [6] showed that even determining feasibility is NP-hard for disc robots, although the proof relies on the robots having different radii.

When considering the optimality of a coordinated motion, one could adopt several different cost measures. Algorithms that optimize motions with respect to total distance traced by the centers of the two robots, the measure adopted in this paper, are typically numerical or iterative in nature, with no precise performance bounds [7]. If the robots are velocity constrained, another natural measure would be realization time [1].

Note that the coordination required to minimize distance is not the same as that required to minimize time.

This paper makes several novel contributions to the understanding of minimum-length coordinated motions. We first characterize all configurations that admit straight-line optimal motions. For all other configurations, the motion from initial to final configuration involves either a net clockwise or counter-clockwise turn in the relative position of the discs. In this case, our results describe either (i) a single optimal motion, or (ii) two feasible motions, of which one is optimal among all net clockwise motions and the other is optimal among all net counter-clockwise motions. For (ii), one of the two motions will be globally optimal. The motions we describe are composed of a constant number of straight segments and circular arcs, of radius s , the sum of the disc radii. The total path length can be expressed as a simple integral depending only on the initial and final positions of the discs.

Our general approach is based on the Cauchy surface area formula, which was first applied to motion planning by Icking et al. [3] to establish the optimality of motions of a directed line segment in the plane, where distance is measured by the length sum of the paths traced by the two endpoints of the segment. (The problem of optimizing the motion of a line segment, even in the absence of obstacles, has a rich history; see [3], and references therein.) Of course, the problem of moving a directed line segment of length s corresponds exactly to the coordinated motion of two discs with radius sum s constrained to remain in contact throughout the motion. Hence the coordinated motion of two discs with radius sum s is equivalent to the problem of moving an “extensible” line segment that can extend freely but has minimum length s . As such, our results also generalize those of Icking et al. [3]. Although we use some of the same tools introduced by Icking et al., our generalization is non-trivial; the path-embedding argument that lies at the heart of the proof of Icking et al. depends in an essential way on the assumption that the rod length is fixed throughout the motion.

The rest of the paper is organized as follows. In Section 2 we outline some basic definitions. Section 3 summarizes the general structure of our proofs, and the main proof and algorithm is presented in Section 4. It is the (unavoidable) nature of our results that they involve an extensive case analysis, far more than can be dealt with comprehensively in six pages. Our objective is to

*Department of Computer Science, UBC, kirk@cs.ubc.ca

†Department of Computer Science, UBC, paul.liu.ubc@gmail.com

provide sufficient detail on a small number of typical cases to convey the general nature of our constructions, together with indications of where extraordinary treatment is required. The appendix provides more comprehensive proofs, accompanied by figures that capture the case analysis in a graphical fashion; full details will appear in the forthcoming MSc thesis of the second author.

2 Background

We want to describe collision-free coordinated motions, for a pair of disc robots, between their initial and final configurations. We specify the (instantaneous) **position** of a disc by the location (in \mathfrak{R}^2) of its centre. A **placement** of a disc pair (\mathbb{A}, \mathbb{B}) is a pair (A, B) , where A (resp. B) denotes the position of \mathbb{A} (resp. \mathbb{B}). A placement (A, B) is said to be **compatible** if $\|A - B\| \geq s$.

A **trajectory** $\xi_{\mathbb{A}}$ of a disc \mathbb{A} from a position A_0 to a position A_1 is a continuous, rectifiable curve $\xi_{\mathbb{A}} : [0, 1] \rightarrow \mathfrak{R}^2$, where $\xi_{\mathbb{A}}(0) = A_0$, $\xi_{\mathbb{A}}(1) = A_1$. The length $\ell(\xi_{\mathbb{A}})$ of a trajectory $\xi_{\mathbb{A}}$ is simply the arc-length of its trace.

A **(coordinated) motion** m of a disc pair (\mathbb{A}, \mathbb{B}) from a placement (A_0, B_0) to a placement (A_1, B_1) is a pair $(\xi_{\mathbb{A}}, \xi_{\mathbb{B}})$, where $\xi_{\mathbb{A}}$ (resp. $\xi_{\mathbb{B}}$) is a trajectory of \mathbb{A} (resp. \mathbb{B}) from position A_0 to a position A_1 (resp. position B_0 to a position B_1). A motion is said to be **compatible** if all of its associated placements are compatible. Unless otherwise specified, all placements and motions that arise in this paper are compatible.

The **separation** between two placements (A_0, B_0) and (A_1, B_1) is simply $\|A_1 - A_0\| + \|B_1 - B_0\|$, the net Euclidean distance traversed by discs \mathbb{A} and \mathbb{B} in moving from placement (A_0, B_0) to placement (A_1, B_1) . The separation clearly provides a lower bound on the **length** $\ell(m)$ of any motion m between placements (A_0, B_0) and (A_1, B_1) , defined as the sum of the lengths of its associated trajectories. Finally, the **(collision-free) distance** $d(P_0, P_1)$ between two placements $P_0 = (A_0, B_0)$ and $P_1 = (A_1, B_1)$ is the minimum possible length over all compatible motions m from P_0 to P_1 . We refer to any compatible motion m between P_0 and P_1 satisfying $\ell(m) = d(P_0, P_1)$ as an **optimal** motion from P_0 to P_1 .

3 The general approach

Suppose that the discs have initial placement $P_0 = (A_0, B_0)$ and final placement $P_1 = (A_1, B_1)$, and let $m = (\xi_{\mathbb{A}}, \xi_{\mathbb{B}})$ be any motion from P_0 to P_1 . Denote by $\widehat{\xi_{\mathbb{A}}}$ (resp. $\widehat{\xi_{\mathbb{B}}}$) the closed curve defining the boundary of the convex hull of $\xi_{\mathbb{A}}$ (resp. $\xi_{\mathbb{B}}$). Since $\xi_{\mathbb{A}}$ (resp. $\xi_{\mathbb{B}}$), together with the segment $\overline{A_0A_1}$ (resp. $\overline{B_0B_1}$), forms a closed curve whose convex hull has boundary $\widehat{\xi_{\mathbb{A}}}$ (resp.

$\widehat{\xi_{\mathbb{B}}}$), it follows from convexity that:

$$\ell(\xi_{\mathbb{A}}) \geq \ell(\widehat{\xi_{\mathbb{A}}}) - \ell(\overline{A_0A_1}) \quad \text{and} \quad \ell(\xi_{\mathbb{B}}) \geq \ell(\widehat{\xi_{\mathbb{B}}}) - \ell(\overline{B_0B_1}) \quad (1)$$

Given a placement $P = (A, B)$, we refer to the angle formed by the vector from B to A with respect to the x -axis as the *angle* of the placement P . Let $[\theta_0, \theta_1]$ be the range of angles counter-clockwise between the angle of P_0 and P_1 .

Observation 1 *Let m be any motion from P_0 to P_1 , and let I denote the range of angles realized by the set of placements in m . Then $[\theta_0, \theta_1] \subseteq I$ or $S^1 - [\theta_0, \theta_1] \subseteq I$.*

We use Observation 1 to categorize the motions we describe into *net clockwise* and *net counter-clockwise* motions. Net clockwise motions satisfy $S^1 - [\theta_0, \theta_1] \subseteq I$ and net counter-clockwise motions satisfy $[\theta_0, \theta_1] \subseteq I$.

Since any motion is either net clockwise or net counter-clockwise (or both) it suffices to optimize over net clockwise and net counter-clockwise motions separately. The following lemma sets out sufficient conditions for net (counter-)clockwise motions to be optimal.

Lemma 1 *Let $m = (\xi_{\mathbb{A}}, \xi_{\mathbb{B}})$ be any net (counter-)clockwise motion from P_0 to P_1 satisfying the following properties:*

1. $\widehat{\xi_{\mathbb{A}}} = \xi_{\mathbb{A}} \cup \overline{A_0A_1}$ and $\widehat{\xi_{\mathbb{B}}} = \xi_{\mathbb{B}} \cup \overline{B_0B_1}$; and
2. $\ell(\widehat{\xi_{\mathbb{A}}}) + \ell(\widehat{\xi_{\mathbb{B}}})$ is minimized over all possible net (counter-)clockwise motions.

Then m is a shortest net (counter-)clockwise motion from P_0 to P_1 .

Proof. Let $m' = (\xi'_{\mathbb{A}}, \xi'_{\mathbb{B}})$ be any net (counter-)clockwise motion from P_0 to P_1 . It follows from property 1 that $\ell(m) = \ell(\widehat{\xi_{\mathbb{A}}}) - \ell(\overline{A_0A_1}) + \ell(\widehat{\xi_{\mathbb{B}}}) - \ell(\overline{B_0B_1})$. Furthermore, from 2 we know that $\ell(\widehat{\xi_{\mathbb{A}}}) + \ell(\widehat{\xi_{\mathbb{B}}}) \leq \ell(\widehat{\xi'_{\mathbb{A}}}) + \ell(\widehat{\xi'_{\mathbb{B}}})$. Thus, using inequality (1), we have

$$\ell(m) \leq \ell(\widehat{\xi'_{\mathbb{A}}}) - \ell(\overline{A_0A_1}) + \ell(\widehat{\xi'_{\mathbb{B}}}) - \ell(\overline{B_0B_1}) \leq \ell(m'). \quad \square$$

When a net (counter-)clockwise motion satisfies the two properties of Lemma 1, we say it is *(counter-)clockwise optimal*. Figure 1 illustrates two motions from the placement (A_0, B_0) to the placement (A_1, B_1) . The blue motion, where \mathbb{B} first pivots about A_0 and moves to B_1 , followed by \mathbb{A} moving from A_0 to A_1 , is counter-clockwise optimal. The yellow motion, where \mathbb{A} first pivots about B_0 and moves to A_1 , followed by \mathbb{B} moving from B_0 to B_1 , is clockwise optimal (as one can check following the proofs of Section 4). However, only the yellow motion is globally optimal.

While property 1 of Lemma 1 is typically easy to verify, property 2 is less straightforward. Fortunately, an application of Cauchy's surface area formula (Theorem

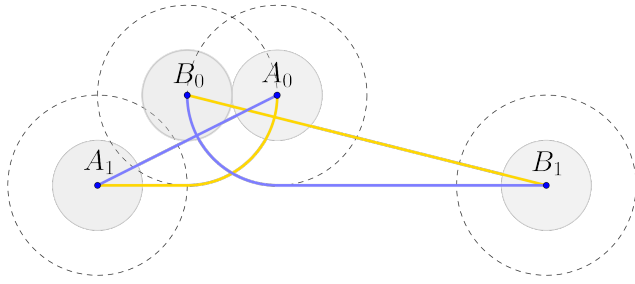


Figure 1: Clockwise (yellow) and counter-clockwise (blue) motions satisfying the two properties of Lemma 1.

2) suffices in all cases of interest. Theorem 2 allows us to translate the problem of measuring lengths of curves into a problem of measuring the support functions of $\widehat{\xi}_A$ and $\widehat{\xi}_B$ at certain critical angles. As it turns out, checking the properties of a curve’s support function is much easier than those of its length.

Definition 1 Let C be a closed curve. The **support function** $h_C : S^1 \rightarrow \mathbb{R}$ of C is defined as

$$h_C(\theta) = \sup\{x \cos \theta + y \sin \theta : (x, y) \in C\}.$$

For an angle θ , the set points that realize the supremum above are called **support points**, and the line oriented at angle $\frac{\pi}{2} + \theta$ going through the support points is called the **support line** (see Figure 2).

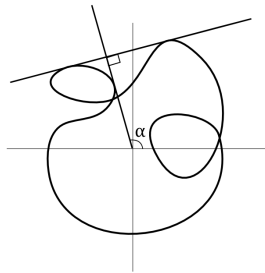


Figure 2: The (two) support points and support line at angle α of a given curve.

Theorem 2 (Cauchy’s surface area formula [2, Section 5.3]) Let C be a closed convex curve in the plane and h_C be the support function of C . Then

$$\ell(C) = \int_0^{2\pi} h_C(\theta) d\theta. \quad (2)$$

As noted in [3], it follows from Theorem 2 that we can bound the length of two convex curves in the plane:

Corollary 3 Let C_1 and C_2 be closed convex curves in the plane. Then the sum of their lengths can be expressed as follows:

$$\ell(C_1) + \ell(C_2) = \int_0^{2\pi} (h_1(\theta) + h_2(\pi + \theta)) d\theta, \quad (3)$$

where h_i is the support function of C_i .

In order to argue the optimality of our motions, we use the following observation that provides a bound on support functions in specified ranges. Let h_A (resp. h_B) denote the support function of $\widehat{\xi}_A$ (resp. $\widehat{\xi}_B$), and let $h_{AB}(\theta)$ denote the sum $h_A(\theta) + h_B(\pi + \theta)$. Recall that s is the radii sum of the two disks.

Observation 2 Let P_0 and P_1 be two configurations and let $[\theta_0, \theta_1]$ be the range of angles counter-clockwise between the angles of P_0 and P_1 . Then, for all net counter-clockwise motions from P_0 to P_1 , and $\theta \in [\theta_0, \theta_1]$, $h_{AB}(\theta) \geq s$. Similarly, for all net clockwise motions and $\theta \in S^1 - [\theta_0, \theta_1]$, $h_{AB}(\theta) \geq s$.

For all support angles, the support function h_A (resp. h_B) is lower bounded by the support function H_A (resp. H_B) of $\overline{A_0A_1}$ (resp. $\overline{B_0B_1}$), since $\overline{A_0A_1} \subset \widehat{\xi}_A$ (resp. $\overline{B_0B_1} \subset \widehat{\xi}_B$). For all of our specified motions, it will be the case that whenever this lower bound is not achieved, the one given by Observation 2 is.

In the next section we give explicit constructions of optimal motions for many initial-final configuration pairs. This includes, of course, all those whose associated trajectories correspond to two straight segments, what we refer to as **straight-line motions**. In other cases, we construct both the clockwise and counter-clockwise optimal motions, one of which must be optimal among all motions.

4 Optimal paths for two discs

Our constructions of shortest (counter-)clockwise motions can be summarized by the following theorem:

Theorem 4 Let A and B be two discs with radius sum s in an obstacle-free plane with arbitrary initial and final placements $P_0 = (A_0, B_0)$ and $P_1 = (A_1, B_1)$. Then there is a shortest motion from P_0 to P_1 composed of at most three circular arcs of radius s and straight segments.

We devote this entire section to the identification and exhaustively treatment of various cases of Theorem 4. The paths that we identify in each case also allow us to provide the following unified characterization of the optimal path length, covering all cases:

Corollary 5 Let $H_{\mathbb{A}}$ and $H_{\mathbb{B}}$ be the support functions of the segments $\overline{A_0A_1}$ and $\overline{B_0B_1}$ respectively, $H_{\mathbb{A}\mathbb{B}}(\theta) := H_{\mathbb{A}}(\theta) + H_{\mathbb{B}}(\pi + \theta)$, and m be an optimal motion between P_0 and P_1 . Let $[\theta_0, \theta_1]$ be the range of angles counter-clockwise between P_0 and P_1 . Then

$$\ell(m) = \min \left(\int_0^{2\pi} \max(H_{\mathbb{A}\mathbb{B}}(\theta), s \cdot \mathbb{1}_{[\theta_0, \theta_1]}) d\theta, \int_0^{2\pi} \max(H_{\mathbb{A}\mathbb{B}}(\theta), s \cdot \mathbb{1}_{S^1 - [\theta_0, \theta_1]}) d\theta \right) - |\overline{A_0A_1}| - |\overline{B_0B_1}|$$

where $\mathbb{1}_{[a,b]}$ is the indicator function of the interval $[a, b]$.

Though the expression in Corollary 5 looks daunting, the only difference between the two integrals is the indicator function used. The support functions themselves can be expressed in closed form and the integrals are clearly lower bounds on the path length by Corollary 3 and Observation 2. We emphasize that the integrals can be expressed in closed form if needed, albeit with some cases involved.

We now introduce some additional tools that will help us classify the initial and final placements into different cases.

Let p and q be arbitrary points. We denote by $\text{circ}_s(p)$ the open disk of radius s centred at point p . In addition we use (i) $\text{corr}_s(p, q)$ to denote the corridor formed by the Minkowski sum of the line segment \overline{pq} and an open disk of radius s , and (ii) $\text{cone}_s(p, q)$ to denote the cone formed by all half-lines from p that intersect $\text{circ}_s(q)$.

If disc \mathbb{A} is centred at location A then $\text{circ}_s(A)$ corresponds to the locations forbidden to the centre of disc \mathbb{B} in a compatible placement. Note that, if point $A \notin \text{corr}_s(B_0, B_1)$ it is possible to translate \mathbb{B} from B_0 to B_1 without intersecting disc \mathbb{A} with centre at point A .

What follows is a case analysis of various scenarios for the initial and final placements. We first classify the cases by the containment of A_0, A_1, B_0, B_1 within $\text{corr}_s(A_0, A_1)$ and $\text{corr}_s(B_0, B_1)$ (see Table 1). While there appears to be 16 cases—since each point is either contained within a corridor or not—they cluster into just three disjoint collections, referred to as Cases 1, 2 and 3. These are further reduced by symmetries which include (i) interchanging the initial and final placements and (ii) switching the roles of \mathbb{A} and \mathbb{B} .

In all cases our specified motion has a common form—with a possible switch of the roles of \mathbb{A} and \mathbb{B} . We identify an intermediate position A_{int} (possibly A_0 or A_1) and perform the following sequence of (locally shortest) moves:

1. Move \mathbb{A} from A_0 to A_{int} , avoiding $\text{circ}_s(B_0)$;
2. Move \mathbb{B} from B_0 to B_1 , avoiding $\text{circ}_s(A_{\text{int}})$; then

Case	$A_0 \in \text{corr}_s(B_0, B_1)$	$A_1 \in \text{corr}_s(B_0, B_1)$	$B_0 \in \text{corr}_s(A_0, A_1)$	$B_1 \in \text{corr}_s(A_0, A_1)$
1a	false	*	*	false
1b	*	false	false	*
2a	true	*	true	*
2b	*	true	*	true
3a	true	true	false	false
3b	false	false	true	true

Table 1: All cases of possible motions. The * entries mean that the specified condition could be true or false.

3. Move \mathbb{A} from A_{int} to A_1 , while avoiding $\text{circ}_s(B_1)$.

Without loss of generality, assume that our initial and final configurations have been *normalized* as follows: B_0 and B_1 lie on the x -axis with B_0 at the origin, B_1 right of B_0 . In all but a few special cases, we will only examine motions that are net counter-clockwise; net clockwise optimal motions can be obtained by reflecting the initial and final placements across the x -axis and then examining net counter-clockwise motions.

The net counter-clockwise orientation of our proposed motion $m = (\xi_{\mathbb{A}}, \xi_{\mathbb{B}})$ as well as the convexity of $\widehat{\xi}_{\mathbb{A}}$ and $\widehat{\xi}_{\mathbb{B}}$ will typically be straightforward to verify. Hence the bulk of our arguments will use Corollary 3 to show that $\ell(\widehat{\xi}_{\mathbb{A}}) + \ell(\widehat{\xi}_{\mathbb{B}})$ is minimized.

4.1 Case 1

It suffices to treat Case 1a, as Case 1b reduces to Case 1a by either symmetry (i) or (ii). In Case 1a, $A_0 \notin \text{corr}_s(B_0, B_1)$, so on the first step we translate \mathbb{B} from B_0 to B_1 in a straight line without touching \mathbb{A} . At this point \mathbb{A} can move freely in a straight line from A_0 to A_1 , as $B_1 \notin \text{corr}_s(A_0, A_1)$. As we shall see through examining the other cases, Case 1 is the only situation where a straight-line motion is possible.

4.2 Case 2

It suffices to treat Case 2a since Case 2b reduces to Case 2a by symmetry (i); thus we assume that $A_0 \in \text{corr}_s(B_0, B_1)$ and $B_0 \in \text{corr}_s(A_0, A_1)$.

There are many subcases to consider, depending on the location of A_1 and B_1 relative to A_0 . We start with the simplest of these, those that arise when $\text{cone}_s(A_0, B_0)$ and $\text{circ}_s(B_1)$ are disjoint (see Figure 3).

4.2.1 $\text{cone}_s(A_0, B_0)$ and $\text{circ}_s(B_1)$ do not intersect

In this case, one can construct zones I-IV in Figure 3 in which each zone has a net counter-clockwise optimal motion of a different form. The zones are constructed through the following tangents and curves:

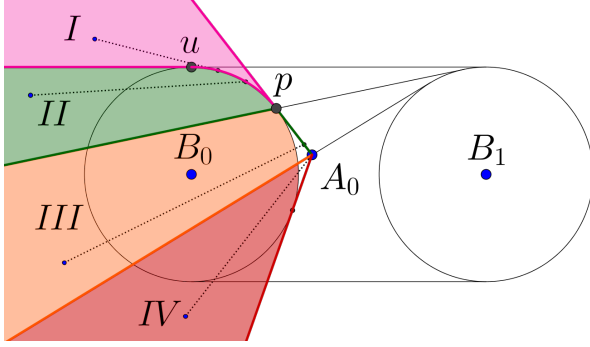


Figure 3: Zones of Case 2 (Section 4.2). Typical paths in each zone are shown as dotted lines.

1. The horizontal tangent through the uppermost point u of $\text{circ}_s(B_0)$. This tangent and the arc of $\text{circ}_s(B_0)$ between u and p (where p is the upper tangent point between A_0 and $\text{circ}_s(B_0)$) separates zone I and II.
2. The tangent through p to $\text{circ}_s(B_1)$. This tangent separates zone II and III.
3. The tangent line from A_0 to $\text{circ}_s(B_1)$. This tangent separates zone III and IV.

Note that zone III and IV may be empty, if the position of A_0 lies below the line tangent to the bottom of $\text{circ}_s(B_0)$ and the top of $\text{circ}_s(B_1)$.

For each zone we specify the location of the intermediate point A_{int} as follows:

[zone I] A_{int} is the point A_1 .

[zone II] A_{int} is the rightmost point of intersection between the tangent from A_1 to $\text{circ}_s(B_1)$ and $\text{circ}_s(B_0)$.

[zone III] A_{int} the point of intersection of the tangent from A_1 to $\text{circ}_s(B_1)$ and the tangent from A_0 to $\text{circ}_s(B_0)$.

[zone IV] A_{int} is the point A_0 .

We then define points T_0 and T_1 which are the lower points of tangency to $\text{circ}_s(A_{\text{int}})$ from B_0 and B_1 respectively. Our three-step generic motion involves:

1. Moving \mathbb{A} on the shortest path from A_0 to A_{int} , avoiding $\text{circ}_s(B_0)$. This may involve rotating \mathbb{A} counter-clockwise about B_0 in a range of angles $[\alpha_0, \alpha_1]$.
2. Moving \mathbb{B} from B_0 to B_1 avoiding $\text{circ}_s(A_{\text{int}})$. This involves translating \mathbb{B} from B_0 to T_0 , rotating \mathbb{B} counter-clockwise about A_{int} from T_0 to T_1 in a range of angles $[\beta_0, \beta_1]$, and then translating \mathbb{B} from T_1 to B_1 .
3. Translating \mathbb{A} from A_{int} to A_1 (collision-free by the disjointness of $\text{cone}_s(A_0, B_0)$ and $\text{circ}_s(B_1)$).

Claim 1 *The motions described above are optimal.*

Proof. It is easy to check that property 1 of Lemma 1 is satisfied. To show that property 2 holds as well we verify that $h_{\mathbb{A}\mathbb{B}}(\theta)$ matches its lower bound and then use Corollary 3. We check that, for all angles θ , either $h_{\mathbb{A}\mathbb{B}}(\theta) = s$ or is determined by \mathbb{A} and \mathbb{B} in their initial or final position.

If the range of angles $[\alpha_0, \alpha_1]$ is non-empty then, by construction, α_0 is normal to the right tangent from A_0 to $\text{circ}_s(B_0)$ and α_1 is normal to the left tangent from A_{int} to $\text{circ}_s(B_0)$. This ensures that for the range of angles $[\alpha_0, \alpha_1]$, B_0 is a support point of $h_{\mathbb{B}}(\theta + \pi)$ while the support point of $h_{\mathbb{A}}(\theta)$ lies on the arc of the circle traversed by \mathbb{A} . Hence $h_{\mathbb{A}\mathbb{B}}(\theta) = s$ for $\theta \in [\alpha_0, \alpha_1]$.

Similarly, if the range of angles $[\beta_0, \beta_1]$ is non-empty then, by construction, β_0 is normal to the right tangent from B_0 to $\text{circ}_s(A_{\text{int}})$ and β_1 is normal to the left tangent from B_1 to $\text{circ}_s(A_{\text{int}})$ (equivalently, the left tangent from A_{int} to $\text{circ}_s(B_1)$). This ensures that for the range of angles $[\beta_0, \beta_1]$, A_{int} is the support point of $h_{\mathbb{A}}(\theta)$ while the support point of $h_{\mathbb{B}}(\theta + \pi)$ lies on the arc of the circle traversed by \mathbb{B} . Hence $h_{\mathbb{A}\mathbb{B}}(\theta) = s$ for $\theta \in [\beta_0, \beta_1]$.

For angles in $S^1 - [\beta_0, \beta_1] - [\alpha_0, \alpha_1]$, it is easy to confirm that either A_0 or A_1 must be one support point, and either B_0 or B_1 must be the other. \square

Note that if A_1 is replaced by any point on the segment joining A_{int} and A_1 the optimal motion is unchanged, except for a shortening of the third move. Consequently, although they do not fall under case 2 according to our classification, configurations in which A_1 lies in the small white triangle at the apex of $\text{cone}_s(A_0, B_0)$ (see Figure 3) are completely covered by the analysis above.

4.2.2 $\text{cone}_s(A_0, B_0)$ and $\text{circ}_s(B_1)$ intersect

Although conceptually very similar (for the most part), the optimal motions that arise when $\text{cone}_s(A_0, B_0)$ and $\text{circ}_s(B_1)$ intersect are complicated by the fact that the path from A_{int} to A_1 must avoid $\text{circ}_s(B_1)$. In particular, placements of A_{int} inside $\text{circ}_s(B_1)$ are forbidden. (Figures 7, 8, and 9 illustrate the modified zones and associated paths. See the appendix for details on the optimality arguments for these paths.)

A more serious complication arises when A_0 occupies an extreme position in the bottom of $\text{corr}_s(B_0, B_1)$ (see Fig. 7). In this situation, when A_1 lies within both $\text{cone}_s(A_0, B_0)$ and $\text{cone}_s(A_0, B_1)$, the motion suggested by our previous zone I analysis does not satisfy the properties needed to support a proof of optimality using Lemma 1. Fortunately, this corresponds to a special case of a more general scenario in which we can prove that there exists a clockwise-optimal motion that is globally optimal (making it unnecessary to try and construct a counter-clockwise optimal motion). Fur-

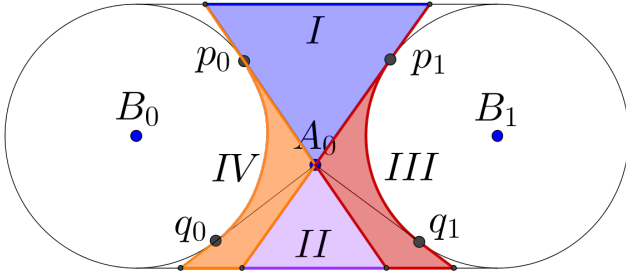


Figure 4: Zones of Case 3 (Section 4.3). Typical paths in each zone are shown as dotted lines.

thermore, the clockwise motion will correspond to a vertically reflected version of a counter-clockwise motion that we've proven optimal above. A technical lemma (Lemma 6) describing this scenario is included in the appendix.

4.3 Case 3

Given the results of our Case 1 and Case 2 analysis (and its symmetric counterpart when the initial and final configurations are interchanged), we can assume that (i) both A_0 and A_1 lie in $\text{corr}_s(B_0, B_1)$.

After removing cases in which the motion is counter-clockwise optimal (by Lemma 6), Case 3 becomes highly constrained. We focus on situations where the motion is not known to be clockwise optimal.

4.3.1 Counter-clockwise optimal motions

Figure 4 illustrates the zone definition for all cases where the optimal motion is not known to be net-clockwise. The zones I-IV of Figure 4 are defined by the following curves:

1. The two upper tangents from A_0 to $\text{circ}_s(B_0)$ and $\text{circ}_s(B_1)$ (through tangent points p_i). These tangents separate zone I from the rest of the zones. The tangent from A_0 to p_1 forms the left boundary of zone II if A_0 is below the tangent from the bottom of $\text{circ}_s(B_0)$ to the top of $\text{circ}_s(B_1)$. The tangent from A_0 to p_0 forms part of the right boundary of zone II.
2. The two horizontal tangents from $\text{circ}_s(B_0)$.
3. The lower tangent from A_0 to $\text{circ}_s(B_0)$ and $\text{circ}_s(B_1)$ (through tangent points q_i). The tangent from A_0 to q_1 (resp. q_0) form part of the right (resp. left) boundary for zone III (resp. zone IV). The tangent from A_0 to q_0 (resp q_1) forms the left (resp. right) boundary of zone II if A_0 is above the tangent from below $\text{circ}_s(B_0)$ to above $\text{circ}_s(B_1)$ (resp. above $\text{circ}_s(B_0)$ to below $\text{circ}_s(B_1)$).

4. The arc of $\text{circ}_s(B_0)$ (resp. $\text{circ}_s(B_1)$) from p_0 to t_0 (resp. p_1 to t_1). If the tangent from A_0 to p_0 (resp. to p_1) does not intersect $\text{circ}_s(B_1)$ (resp. $\text{circ}_s(B_0)$), then t_0 is q_0 (resp. t_1 is q_1). Otherwise, t_0 (resp. t_1) is the intersection point.
5. The arc of $\text{circ}_s(B_0)$ (resp. $\text{circ}_s(B_1)$) from t_0 to q_0 (resp. t_1 to q_1). These arcs forms part of the left and right boundaries of zone II.

We now specify, for each zone, the location of A_{int} , and define T_0 and T_1 to be the lower tangent points of B_0 and B_1 to $\text{circ}_s(A_{\text{int}})$ respectively.

[zone I] A_{int} is the point A_1 .

[zone II] A_{int} is the point A_0 .

[zone III] A_{int} is the intersection point of the tangent from A_1 to the $\text{circ}_s(B_0)$ and the tangent from A_0 to $\text{circ}_s(B_1)$.

[zone IV] A_{int} is the intersection point of the tangent from A_0 to the $\text{circ}_s(B_0)$ and the tangent from A_1 to $\text{circ}_s(B_1)$.

Our generic three-stage motion then becomes:

1. Move \mathbb{A} on a straight line from A_0 to A_{int}
2. Move \mathbb{B} from B_0 to B_1 avoiding $\text{circ}_s(A_{\text{int}})$. This involves moving \mathbb{B} to T_0 , rotating \mathbb{B} counter-clockwise about A_0 to T_1 in a range of angles $[\beta_0, \beta_1]$, and then moving \mathbb{B} from T_1 to B_1 .
3. Move \mathbb{A} on a straight line motion from A_{int} to A_1 .

Note that in zone IV of Figure 4, all optimal counter-clockwise motions are of exactly the same form as zone III of Case 2.

Claim 2 *The motions described above are optimal.*

Proof. As before, it is easy to check that property 1 of Lemma 1 is satisfied. To show that property 2 holds as well we verify that $h_{\mathbb{A}\mathbb{B}}(\theta)$ matches its lower bound and then use Corollary 3. We check that, for all directions θ , either $h_{\mathbb{A}\mathbb{B}}(\theta) = s$ or the support function is determined by \mathbb{A} and \mathbb{B} in their initial or final position.

By construction, β_0 is normal to the right tangent from B_0 to $\text{circ}_s(A_{\text{int}})$ (equivalently, the right tangent from A_{int} to $\text{circ}_s(B_0)$) and β_1 is normal to the left tangent from B_1 to $\text{circ}_s(A_{\text{int}})$ (equivalently, the left tangent from A_{int} to $\text{circ}_s(B_1)$). This ensures that for the range of angles $[\beta_0, \beta_1]$, A_{int} is one support point while the other support point lies on the arc of the circle traversed by \mathbb{B} . Hence $h_{\mathbb{A}\mathbb{B}}(\theta) = s$ for $\theta \in [\beta_0, \beta_1]$.

For angles in $S^1 - [\beta_0, \beta_1]$, it is easy to confirm that either A_0 or A_1 must be one support point, and either B_0 or B_1 must be the other. \square

In Figures 10 and 11 of the appendix, we show how some exceptional cases are handled by Lemma 6, such as the case where the circles are intersecting.

5 Angle monotone motions

In the sections above, we've stated all of our motions as *decoupled* motions where only one of \mathbb{A} or \mathbb{B} is moving at a time. However, we can produce angle monotone motions by simply *coupling* the motions given. The coupling itself simply corresponds to following the trace of motions in the previous sections while keeping \mathbb{A} and \mathbb{B} as close together as possible. Furthermore, one can check that for all of our optimal motions, our coupling ensures that \mathbb{A} and \mathbb{B} are in contact for a connected interval of time.

As far as we know, our tools are limited to the case when the robots are discs in 2D. Indeed, when the robots are spheres in 3D, even if the initial and final positions of the robot are coplanar, we are unsure if the shortest path stays within the plane (except in special cases). The 3D extension of the problem as well as the 2D problem with obstacles remain subjects for future exploration.

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Appendix

5.1 Clockwise-minimal motions

We now describe a set of placements for which we can prove that the optimal motion is net clockwise. This will help us deal with subcases within Case 2 and Case 3 for which the demonstration of net counter-clockwise optimal motions seems to be beyond the reach of techniques based on Cauchy’s surface area formula.

Lemma 6 *Suppose that $A_0, A_1 \in \text{corr}_s(B_0, B_1)$ and $\text{cone}_s(A_i, B_i)$ intersects $\text{circ}_s(B_j)$ for some $i, j \in \{0, 1\}$ with $i \neq j$. Let H_i be the half-space under the tangent of $\text{cone}_s(A_i, B_i)$ intersecting $\text{circ}_s(B_j)$ and suppose that $A_j \in H_i$. If A_i is below the line connecting B_0 with B_1 , then the optimal motion must be net clockwise. Otherwise the optimal motion is net counter-clockwise.*

Proof. There are two major cases: (i) the case where $\text{circ}_s(B_0)$ does not intersect $\text{circ}_s(B_1)$ and (ii) the case where they do intersect. We first deal with case (i).

For both cases, we assume that A_0 is under the line connecting B_0 with B_1 and that $\text{cone}_s(A_0, B_0)$ intersects $\text{circ}_s(B_1)$. The other cases are treated similarly with almost exactly the same proof.

Let U_1 be the upper tangent point of A_0 to $\text{circ}_s(B_1)$. By our assumptions, $\text{cone}_s(A_0, B_0)$ intersecting $\text{circ}_s(B_1)$ implies that $\text{cone}_s(A_0, B_1)$ intersects $\text{circ}_s(B_0)$. Hence A_1 lies below $\overline{A_0U_1}$, and $A_1 \in \text{corr}_s(B_0, B_1)$.

Let V_0 be the upper tangent point of A_1 to $\text{circ}_s(B_0)$. We first deal with the case where the tangent segments $\overline{A_0U_1}$ and $\overline{A_1V_0}$ intersect at a point $A_{\text{int}} \in \text{corr}_s(B_0, B_1)$ (see Figure 5).

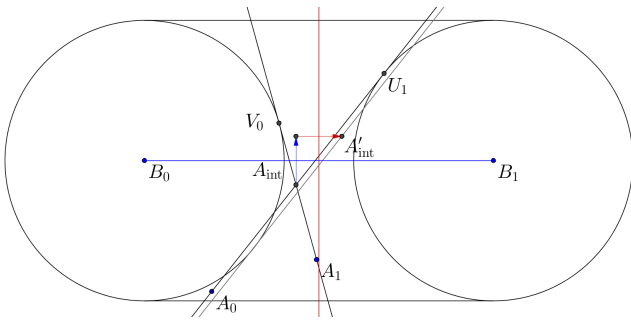


Figure 5: A case of Lemma 6.

Consider the following “motion” $m' = (\xi'_A, \xi'_B)$:

1. Move \mathbb{A} on a straight line from A_0 to A_{int} .
2. Move \mathbb{B} from B_0 to B_1 avoiding $\text{circ}_s(A_{\text{int}})$. This involves moving \mathbb{B} to T_0 (the lower tangent point of B_0 and $\text{circ}_s(A_{\text{int}})$), rotating \mathbb{B} counter-clockwise about A_{int} to T_1 (the lower tangent point of B_1 and

$\text{circ}_s(A_{\text{int}})$) in a range of angles $[\beta_0, \beta_1]$, and then moving \mathbb{B} from T_1 to B_1 .

3. Move \mathbb{A} in a straight line from A_{int} to A_1 .

The “motion” outlined above is impossible, as the position of B_0 prevents the movement from A_0 to A_{int} in a straight-line. However, we show that $\ell(m')$ forms a lower bound on all possible net clockwise motions. Our approach will be to first show that $\ell(\widehat{m}')$ is smaller than $\ell(\widehat{m})$ for any net counter-clockwise motion m and then apply the same reasoning as Lemma 1 to get a lower bound on $\ell(m)$.

Let $h_{\mathbb{A}}$ (resp. $h_{\mathbb{B}}$) be the support function of ξ'_A (resp. ξ'_B) and let $h_{\mathbb{A}\mathbb{B}}$ be their sum. As before, let $[\theta_0, \theta_1]$ be the range of angles counter-clockwise between P_0 and P_1 . For most angles, either A_0 or A_1 must be one support point, and either B_0 or B_1 must be the other. In fact, in a straight-line motion, this is always the case. However, due to the circular arc traversed in step 2 of m' , one support point will be on the arc of the circle traversed by \mathbb{B} for angles in $[\beta_0, \beta_1]$. The other support point will be A_{int} , since the segment $\overline{B_0T_0}$ is parallel to the segment $\overline{A_0A_{\text{int}}}$ and the segment $\overline{B_1T_1}$ is parallel to the segment $\overline{A_1A_{\text{int}}}$. Hence $h_{\mathbb{A}\mathbb{B}}(\theta) = s$ for $\theta \in [\beta_0, \beta_1]$. Note that by construction of m' , $[\beta_0, \beta_1] \subset [\theta_0, \theta_1]$. By Observation 2, $h_{\mathbb{A}\mathbb{B}}(\theta)$ matches its lower bound for $\theta \in [\beta_0, \beta_1]$. For angles in $S^1 - [\beta_0, \beta_1]$, one can see that either A_0 or A_1 must be one support point, and either B_0 or B_1 must be the other. Hence $h_{\mathbb{A}\mathbb{B}}(\theta)$ matches its lower bound for all angles θ .

By Cauchy’s theorem, $\ell(m') = \int_0^{2\pi} h_{\mathbb{A}\mathbb{B}}(\theta)d\theta$. Since $h_{\mathbb{A}\mathbb{B}}$ matches its lower bound, $\ell(m') \leq \ell(m)$ for any net counter-clockwise motion m .

Now we construct a net clockwise motion whose length is no greater than that of m' . Construct the point A'_{int} in Figure 5, which is the result of two reflections of A_{int} , first along the line from B_0 to B_1 and then along the perpendicular bisector of $\overline{B_0B_1}$. Consider the following motion m :

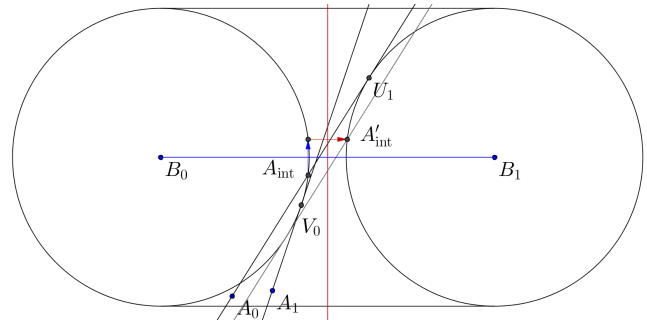


Figure 6: A case of Lemma 6.

1. Move \mathbb{B} from B_0 to B_1 avoiding $\text{circ}_s(A'_{\text{int}})$.

2. Move A_0 to A_1 in a straight line.

Clearly step 1 of m is the same length as step 2 of m' , and step 2 of m is at most the length of steps 1 and 3 of m' , so $\ell(m) \leq \ell(m')$. Furthermore m is a feasible motion. To see this, let t be line through A'_{int} parallel to the segment $\overline{A_0 A_{\text{int}}}$, and let q be the tangent point between $\text{circ}_s(B_0)$ and t . Note that A'_{int} lies on the right of q above t and A_0 is left of q and above t , so A_0 does not obstruct the movement of \mathbb{B} in step 1.

Hence the optimal motion must be net clockwise in the case where $\overline{A_0 U_1}$ and $\overline{A_1 V_0}$ intersect.

When $\overline{A_0 U_1}$ and $\overline{A_1 V_0}$ do not intersect (see Figure 6), this means that V_0 is below $\overline{A_0 U_1}$. In this case, let A_{int} be the right-most intersection point between $\overline{A_0 U_1}$ and $\text{circ}_s(B_0)$ and the proof above will work without modification.

We now deal with case (ii), where $\text{circ}_s(B_0)$ intersects $\text{circ}_s(B_1)$. Let \mathcal{L} (resp. \mathcal{U}) denote the region within $\text{corr}_s(B_0, B_1)$ below (resp. above) the discs enclosed by $\text{circ}_s(B_0)$ and $\text{circ}_s(B_1)$. We will show that if both A_0 and A_1 are in \mathcal{L} , then the optimal motion must be net clockwise. The case for \mathcal{U} can be handled similarly.

As before, we will first lower bound the optimal net counter-clockwise motion by an impossible motion, and then show a net clockwise motion that is at most the length of the lower bound.

Suppose that A_0 is left of the perpendicular bisector of $\overline{B_0 B_1}$. Let U_1 be the upper tangent point of A_0 to $\text{circ}_s(B_1)$. Let V_0 be the upper tangent point of A_1 to $\text{circ}_s(B_0)$. Let t be the upper intersection point of $\text{circ}_s(B_0)$ and $\text{circ}_s(B_1)$. If A_1 is right of the perpendicular bisector, let U_1 be the upper tangent point of A_0 to $\text{circ}_s(B_0)$, and let V_0 be the upper tangent point of A_1 to $\text{circ}_s(B_1)$.

If V_0 is counter-clockwise of t on $\text{circ}_s(B_0)$ or U_1 is clockwise of t on $\text{circ}_s(B_1)$, one can check that the proof of the non-intersecting case works here as well. Otherwise, both U_0 and V_1 are vertically below t .

In this case, consider the following “motion” m' :

1. Move \mathbb{A} on a straight line from A_0 to t . This involves possibly cutting a chord through $\text{circ}_s(B_0)$ and $\text{circ}_s(B_1)$ in a range of angles $[\alpha_0, \alpha_1]$.
2. Move \mathbb{B} from B_0 to B_1 avoiding $\text{circ}_s(t)$.
3. Move \mathbb{A} in a straight line from t to A_1 . This involves possibly cutting a chord through $\text{circ}_s(B_0)$ and $\text{circ}_s(B_1)$ in a range of angles $[\alpha_2, \alpha_3]$.

We note that the m' used here is almost the same as the m' used for the non-intersecting case, using t in place of A_{int} . For the range of angles $[\alpha_0, \alpha_1]$ one support point will be B_0 as $[\alpha_0, \alpha_1] \subset [-\pi/2, \pi/2]$. The other support point will either be t or one of the A_i 's. Note that when the other support point is t , the support function at most s (and strictly less at all but 1 point) as m' cuts through $\text{circ}_s(B_0)$. Hence $h_{\mathbb{A}\mathbb{B}}(\theta)$ is at most

its lower bound for $\theta \in [\alpha_2, \alpha_3]$. Similarly, for the range of angles $[\alpha_2, \alpha_3]$, one support point will be B_1 , and the other support point will either be t or one of the A_i 's. Again, a similar argument shows that $h_{\mathbb{A}\mathbb{B}}(\theta)$ meets its lower bound in $[\alpha_2, \alpha_3]$ as well.

For all other angles, the argument proceeds exactly as in the non-intersecting case. \square

5.2 Case 2: $\text{cone}_s(A_0, B_0)$ and $\text{circ}_s(B_1)$ intersect

5.2.1 Modified zones in Case 2

The only modification to the paths we've seen in Section 4.2 is to zone IV. Previously, the intermediate point A_{int} in zone IV was always the point A_0 . However in exceptional cases, such as when $A_0 \in \text{circ}_s(B_1)$, A_0 is infeasible as an intermediate point. Let t be this intermediate point, which we define below depending on the relative position of B_0, B_1 , and A_0 .

Let p be the tangent point between A_0 and $\text{circ}_s(B_0)$. If $\text{circ}_s(B_0)$ and $\text{circ}_s(B_1)$ intersects, let q be that intersection point. If $A_0 \in \text{circ}_s(B_1)$, let r be the intersection point between $\text{circ}_s(B_1)$ and the line through A_0 and p .

- If q exists and p is clockwise of q on $\text{circ}_s(B_0)$, then let t be q .
- If q exists and p is counter-clockwise of q on $\text{circ}_s(B_0)$, then let t be r if r exists, and let t be A_0 otherwise.

By using this choice of A_{int} for zone IV and taking care of the exceptional configurations in the next section, one can show through similar proofs to those in Section 4.2 that the zone IV motion is net counter-clockwise optimal.

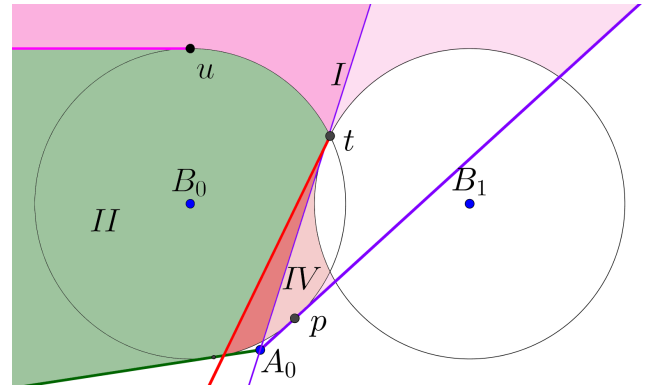


Figure 7: Zone classification when $\text{cone}_s(A_0, B_0)$ and $\text{circ}_s(B_1)$ intersect. In this case we use the tangent to $\text{circ}_s(B_1)$ at t to form the left boundary of zone IV. Regions addressed by Lemma 6 are coloured in lighter shades.

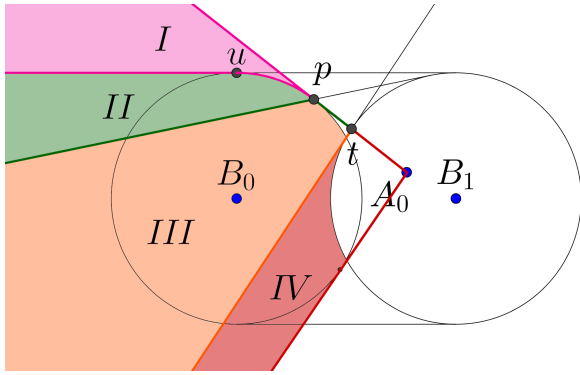


Figure 8: Zone classification when $\text{cone}_s(A_0, B_0)$ and $\text{circ}_s(B_1)$ intersect and $A_0 \in \text{circ}_s(B_1)$.

5.2.2 Exceptional configurations in Case 2

When $\text{cone}_s(A_0, B_0)$ and $\text{circ}_s(B_1)$ intersect, our arguments from Section 4.2 can be insufficient in a variety of ways. For example, when A_0 is as depicted in Figure 7 and A_1 is situated under the lower intersection point of $\text{circ}_s(B_0)$ and $\text{circ}_s(B_1)$, the zone I motion we previously outlined would first move \mathbb{A} from A_0 to A_1 , and then rotate \mathbb{B} from B_0 to B_1 . However, one can verify that this motion does not minimize the support function while B is rotating, as the support point of $h_{\mathbb{A}}$ while B is rotating switches from A_1 to A_0 before the rotation is completely.

Another example can be seen in Figure 9. Suppose A_1 is situated right of $\text{circ}_s(B_1)$ and above the lower horizontal tangent of $\text{circ}_s(B_1)$. Then the zone IV motion previously outlined would first rotate \mathbb{B} from B_0 to B_1 , and then rotate \mathbb{A} from A_0 to A_1 about $\text{circ}_s(B_1)$. However, during this last rotation, the support function is not minimized, as the support point of $h_{\mathbb{B}}$ switches from B_1 to B_0 once \mathbb{A} rotates past $3\pi/2$.

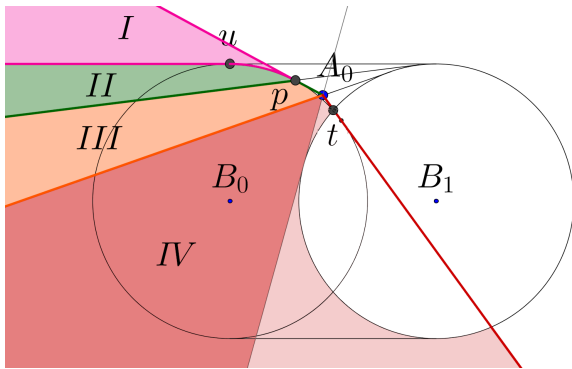


Figure 9: Zone classification when $\text{cone}_s(A_0, B_0)$ and $\text{circ}_s(B_1)$ intersect. Regions addressed by Lemma 6 are coloured in lighter shades.

Fortunately, both of these inefficiencies are addressed by Lemma 6, as in both cases the optimal motion is net clockwise. By an exhaustive treatment, one can verify that the only cases where our Case 2 arguments fail are once for which the optimal motion is net clockwise.

5.2.3 Exceptional configurations in Case 3

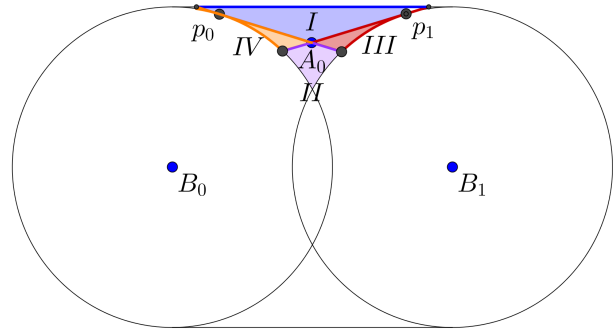


Figure 10: Case 3, when $s\text{-circ}(B_0)$ and $s\text{-circ}(B_1)$ intersect. A_0 and A_1 are both above the intersection.

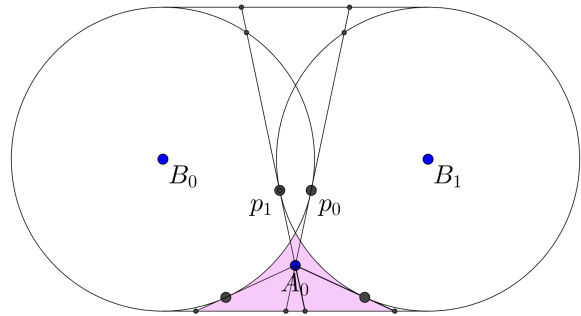


Figure 11: Case 3, when $s\text{-circ}(B_0)$ and $s\text{-circ}(B_1)$ intersect. A_0 and A_1 are both below the intersection.

As stated in Section 4.3, Case 3 is highly constrained, and the zones are simpler than those in Case 2. When $\text{circ}_s(B_0)$ and $\text{circ}_s(B_1)$ intersect, the description of the zones change slightly, and certain configurations become impossible. For example, when $s\text{-circ}(B_0)$ and $s\text{-circ}(B_1)$ intersect, observe that the constraints force either A_0 and A_1 to be both above the B circles, or both below. If one of the A_i 's were above and the other below, then we would be in Case 2. Figures 10 and 11 outline two cases in which the zones differ from our previous description.

In Figure 10, the zones are constructed similarly to Figure 4. However, the lower boundary of zones II-IV is now formed by the boundaries of $\text{circ}_s(B_0)$ and

$\text{circ}_s(B_1)$ above their intersection point. The motions in zones I-IV, as well as proofs of optimality, remain valid for all four zones.

In Figure 11, we note that the optimal motion is clockwise by Lemma 6. After vertically reflecting Figure 11 to obtain the clockwise optimal motion, note that it becomes Figure 10 exactly, and thus is already handled by our previous analysis.