CS369M: Algorithms for Modern Massive Data Set Analysis **Lecture 11 - 10/28/2009**

Expander Graphs in Algorithms Theory and Data Applications (2 of 2)

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**Unedited Notes*

1. Review of basic definitions and results

Definition 1. *Given a graph $G = (V, E)$ and subsets $S, T \subseteq V$, let*

$$E(S,T) = \{(u,v) \in E |\ u \in S,\ v \in T\}$$

Definition 2. *Let the expansion $h(G)$ of a graph $G = (V, E)$ be defined as:*

$$h(G) = min_{S \subseteq V} \frac{|E(S, \bar{S})|}{min(|S|, |\bar{S}|)}$$

Let a d-regular graph G be an expander if $h(G) \geq \epsilon$ where ϵ is a constant. To relate the expansion of a graph with its spectral gap, we have the following theorem due to Cheeger and Buser:

Theorem 1. *If $\mu_0 \geq \mu_1 \geq \ldots \geq \mu_{n-1}$ are the eigenvalues of the adjacency matrix $A(G)$ of d-regular graph G, then:*

$$\frac{d - \mu_1}{2} \leq h(G) \leq \sqrt{2d(d - \mu_1)}$$

Therefore, the expansion of the graph is related to its spectral gap $(d - \mu_1)$. Thus, we can define a graph to be an expander if $\mu_1 \leq \epsilon d$ or $\lambda_1 \geq \epsilon$ where λ_1 is the second eigenvalue of the matrix $L(G) = D - A(G)$ where D is the diagonal degree matrix. Although it is difficult to draw an expander because a constant fraction of the edges necessarily cross, they do exist. In fact, a typical d-regular graph is an expander with high probability under certain random graph models.

One common random graph model is the Erdos-Renyi $\mathcal{G}_{n,p}$ model, where there are n vertices and edges are chosen to exist with probability p. The $\mathcal{G}_{n,m}$ model is another common model where graphs with n vertices and m edges are chosen uniformly at random. If we set p such that there are on average m, edges, then $\mathcal{G}_{n,m}$ is very similar to $\mathcal{G}_{n,p}$ if $p \geq \log n/n$. However, we require in addition that d is a constant. Now if $p = 3/n$, $\mathcal{G}_{n,p}$ usually generates a graph that is not connected. However, $\mathcal{G}_{n,m}$ with corresponding parameters usually yields a connected graph with high expansion. We can think of randomized expander construction as a version of $\mathcal{G}_{n,m}$ further constrained to d-regular graphs. We also note that there exists a deterministic construction for fixed $d/$.

For further intuition on second eigenvalue of the $A(G)$, we note the following:

- For a path graph, $\mu_1 = \Theta(1/n^2)$.
- For a $\sqrt{n} \times \sqrt{n}$ grid, $\mu_1 = \Theta(1/n)$.
- For an expander, $\mu_1 = \Theta(1)$.
- For a $K_n - K_n$ dumbell, $\mu_1 = \Theta(1/n)$.
- For a hypercube, $\mu_1 = \Theta(1/\log n)$.

One natural to ask is how large can the spectral gap be? The following theorem, due to Alon and Boppana bounds this gap:

Theorem 2. *Denoting $\lambda = max(|\mu_1|, |\mu_{n-1}|)$, we have, for every d-regular graph:*

$$\lambda \geq 2\sqrt{d-1} - o_n(1)$$

We note there exists constructions called Ramanujan graphs where the second eigenvalue of $L(G)$, $\lambda_1(G) = d - 2\sqrt{d-1}$. Expanders have a lot of interesting properties. In this lecture, we study some of these properties.

2. Metric Embedding into l_2

It is easy to see that an expander can be embedded into l_2 with distortion $O(\log n)$ (just map the expander nodes to the nodes of a simplex in $\mathbb{R}^n$). We will show that this result is tight.

Recall from the last lecture that for the minimum distortion in embedding a metric space (X, d) into l_2, denoted by $C_2(X, d)$, we have:

$$C_2(X,d) = max_{P \in PSD, P.\overrightarrow{1} = \overrightarrow{0}} \sqrt{\frac{\sum_{p_{ij}>0} p_{ij} d(x_i, x_j)^2}{-\sum_{p_{ij}<0} p_{ij} d(x_i, x_j)^2}} \tag{1}$$

Assume $G = (V, E)$ is a d-regular graph with $\mu_1 \geq d - \epsilon$ and $|V| = n$. Then, we have the following lemma:

Lemma 3. *If $H = (V, E')$ is a graph with the same vertex set as G, in which two vertices u and v are adjacent iff $d_G(u, v) \geq log_d n - 2$, then H has a matching with $n/2$ edges.*

Proof. G is a d-regular graph, hence for any vertex $x \in V$ and any value r, at most d^r vertices $y \in V$ can have $d_G(x, y) \leq r$. Thus, at least half of the nodes of G are further than $log_d n - 2$ from x. This means every node in H has at least degree $n/2$, which proves the lemma.

□

Theorem 4. *If $G = (V, E)$ is a d-regular graph with $\mu_1 \geq d - \epsilon$ and $|V| = n$, then*

$$C_2(G) = \Omega(\log n)$$

Proof. Assume B is the adjacency matrix of the matching proved in the previous lemma. Define $P = (dI - A_G) + \frac{\epsilon}{2}(B - I)$. Then, $P.\overrightarrow{1} = \overrightarrow{0}$, and $P \in$ PSD. This is because, for any $x \perp \overrightarrow{1}$, $x^T(dI - A_G)x \geq \epsilon||x||^2$ (by the assumption on μ_1), and

$$x^T(B - I)x = \sum_{(i,j)\in B} (2x_i x_j - x_i^2 - x_j^2) \geq -\sum (x_i^2 + x_j^2) = 2||x||^2$$

Also, we have:

$$\sum_{p_{ij}>0} d(i,j)^2 p_{ij} \geq \frac{\epsilon}{2} n (log_d n - 2)^2$$

and:

$$-\sum_{p_{ij}<0} d(i,j)^2 p_{ij} = dn$$

Hence, from 1, we have:

$$C_2(G) \geq \Omega(\log n)$$

□

3. Quasirandomness

The following theorem, called "Expander Mixing Lemma", shows that if the spectral gap is large, then the number of edges between two subsets of the graph vertices can be approximated by the same number for a random graph.

Theorem 5. *If $G = (V, E)$ is d-regular, with $|V| = n$ and $\lambda = max(|\mu_1|, |\mu_{n-1}|)$, then for all $S, T \subseteq V$:*

$$||E(S,T)| - \frac{d|S|.|T|}{n}| \leq \lambda\sqrt{|S|.|T|}$$

Proof. Define χ_S and χ_T to be the charactristic vectors of S and T. Then, if $(v_j)_{j=0}^{n-1}$ are orthonormal eigenvectors of A_G, we can write: $\chi_S = \sum \alpha_i v_i$ and $\chi_T = \sum \beta_i v_i$. Thus,

$$|E(S,T)| = \chi_S^T A \chi_T = \sum \mu_i \alpha_i \beta_i$$

$$= d\frac{|S|.|T|}{n} + \sum_{i\geq 1} \mu_i \alpha_i \beta_i$$

where the last inequality is because, $\alpha_0 =< \chi_S, \frac{\overrightarrow{1}}{\sqrt{n}} >= \frac{|S|}{n}$, (similarly) $\beta_0 = \frac{|T|}{n}$, and $\mu_0 = d$.

Hence,

$$||E(S,T)| - d\frac{|S|.|T|}{n}| = |\sum_{i\geq 1} \mu_i \alpha_i \beta_i|$$

$$\leq \sum_{i\geq 1} |\mu_i||\alpha_i||\beta_i| \leq \lambda \sum_{i\geq 1} |\alpha_i||\beta_i|$$

$$\leq \lambda||\alpha||_2||\beta||_2 = \lambda||\chi_S||_2||\chi_T||_2$$

$$= \lambda\sqrt{|S|.|T|}$$

□

4. Random walks on expanders

Random walks on expanders mix rapidly. Assume $G = (V, E)$ with $|V| = n$ is d-regular, A is the adjacency matrix of G, and $\hat{A} = \frac{1}{d}A$ is the transition matrix of a random walk on G. Also, assume $\lambda = max\{|\mu_0|, |\mu_{n-1}|\} = \alpha d$. Then:

Theorem 6. $||\hat{A}^t p - u||_1 \leq \sqrt{n}\alpha^t$, *where u is the stationary distribution of the random walk, and p is an arbitrary distribution on V.*

Proof is given in [1]. This theorem shows that if the spectral gap is large (i.e. α is small), then we the walk mixes rapidly.

References

[1] N. Linial and A. Wigderson, "Expander graphs and their applications"