

Fillomino Densities

Don Knuth, Stanford Computer Science Department
(30 June 2026; revised 01 July 2026)

A “fillomino pattern” is a labeling of grid cells with positive integers in such a way that every cell labeled d is rookwise connected to exactly d cells that have the same label. (Equivalently, it’s a way to pack a shape with polyominoes, where no two d -ominoes have an edge in common.) Here, for example, is a more-or-less random fillomino pattern:

```

33877722
31888777
44488179
33418899
13225999
88855926
81859926
88856666
    
```

Exercise 7.2.2.3–296 of *The Art of Computer Programming* asks about the maximum density of d ’s in any infinite sequence of $n \times n$ fillomino patterns Φ_n , namely

$$\delta_d = \limsup_{n \rightarrow \infty, \text{ patterns } \Phi_n} \frac{\text{the number of } d\text{'s in } \Phi_n}{n^2}.$$

Its answer proves the upper bounds

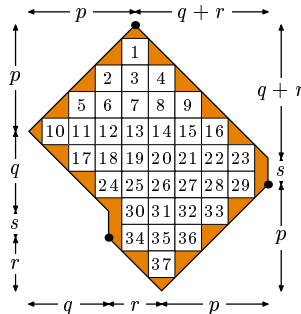
$$\delta_d \leq \frac{d}{d+h_d}, \quad \text{where } h_d = \frac{\lceil \sqrt{8d-4} \rceil}{2},$$

together with lower bounds for $d \geq 3$:

$$\delta_d \geq \begin{cases} d/(b_{2k-1}+k), & \text{if } b_{2k-2} < d \leq b_{2k-1}, \\ d/(b_{2k}+k), & \text{if } b_{2k-1} < d \leq b_{2k}, \end{cases} \quad \text{where } b_t = \left\lfloor \frac{t^2+4}{8} \right\rfloor.$$

Shisheng Li, who wrote to me recently from Singapore [2], has found constructions for $d \leq 25$ that improve those lower bounds significantly. In fact, he has raised them to match the upper bounds exactly, in eleven of the seventeen previously unknown cases!

I happened to notice that eight of those eleven constructions can be unified and generalized, because a certain general family of heptagons is able to tile the plane. These general tiles, which I call $T(p, q, r, s)$ because they are defined by four integer parameters, have a remarkably simple shape, illustrated here in the case $p = 4, q = 3, r = 2,$ and $s = 1$:



The seven vertices lie at the midpoints of pixels. Tiles fit together tightly if, for example, the vertex just left of 34 is moved to the position just above 1 or just right of 29. In general, this heptagon is a generalized torus, where we identify pixels that are shifted by integer multiples of $(p-q, s+q+p)$ and/or $(p+r, p-r)$. Therefore its total area $A(p, q, r, s)$ is $(p+r)(p+q+s) - (p-q)(p-r) = p(2q+2r+s) + rs$. Furthermore, the area of the “halo” at the boundary is $H(p, q, r, s) = p+q+r+s-1$. Hence there are $N(p, q, r, s) = A(p, q, r, s) - H(p, q, r, s)$ pixels in the inner polyomino. (In this example it’s $A(4, 3, 2, 1) - H(4, 3, 2, 1) = 46 - 9 = 37$.)

You may wish to get more familiar with these heptagons by drawing $T(4, 1, 3, 2)$. Curiously, it turns out that $N(4, 1, 3, 2) = N(4, 3, 2, 1)$, although the shapes are quite different.

Almost immediately after I began to play with these heptagonal tiles, I naturally wrote a computer program in order to find out the best bounds that they're able to achieve. I wanted to explore the quantity

$$D(d) = \min_{p,q,r,s \geq 0} \{A(p, q, r, s) \mid N(p, q, r, s) = d\},$$

because we clearly have the lower bound $\delta_d \geq d/D(d)$. The result was a delightful surprise: In every case that I was able to examine, the parameters p , q , r , and s achieved the minimum value for d if and only if $H(p, q, r, s) = \lceil h_d \rceil$.

In other words, it appeared to be true that the heptagons $T(p, q, r, s)$ establish the tight bounds

$$\frac{d}{d + \lceil h_d \rceil} \leq \delta_d \leq \frac{d}{d + h_d}, \quad \text{for all } d \geq 3,$$

with equality if and only if h_d is an integer!

The purpose of this note is to present a rigorous proof of that surprising empirical discovery. The proof below is completely elementary. However, another surprise has been that truly simple methods don't seem to suffice: The result appears to rely on some rather delicate properties of quadratic polynomials.

We shall, in fact, prove a stronger result, by showing that the minimum value $D(d)$ is almost always obtained by a heptagon for which the "vertical jag" parameter s is equal to 1. In this special case we shall write simply $T(p, q, r)$, $A(p, q, r)$, $H(p, q, r)$, and $N(p, q, r)$ instead of $T(p, q, r, s)$, $A(p, q, r, s)$, $H(p, q, r, s)$, and $N(p, q, r, s)$. The new formulas are appealingly simple:

$$\begin{aligned} A(p, q, r) &= p(2q + 2r + 1) + r; \\ H(p, q, r) &= p + q + r; \\ N(p, q, r) &= 2p(q + r) - q. \end{aligned}$$

Theorem. *For all $d \geq 1$ there exist $p, q, r \geq 0$ such that $N(p, q, r) = d$ and $H(p, q, r) = \lceil \sqrt{2d-1} \rceil$, except when $d = b_{4t-2} = 2t^2 - 2t + 1 = (t-1)^2 + t^2$ is the sum of consecutive squares.*

Before discussing the proof, let's observe first that the exceptional case corresponds to a trivial square tile that has been rotated 45° , namely to an instance of $T(p, q, r, s)$ with $p = q$ and $r = s = 0$. Indeed, we have $A(t, t, 0, 0) = 2t^2$ and $H(t, t, 0, 0) = 2t - 1 = \sqrt{2d-1}$ when $d = (t-1)^2 + t^2$, hence $N(t, t, 0, 0) = d$.

Furthermore, we cannot have $N(p, q, r) = d$ and $H(p, q, r) < \lceil \sqrt{2d-1} \rceil$. For the definition of h_d tells us that we'd have $H(p, q, r) < h_d$, contradicting the fact that $d/(d+h_d) \geq \delta_d \geq d/A(p, q, r) = d/(d+H(p, q, r))$.

There's a handy rule for understanding the quantity h_d :

$$h_d = \begin{cases} t - \frac{1}{2}, & \text{if } b_{2t-2} < d \leq b_{2t-1}; \\ t, & \text{if } b_{2t-1} < d \leq b_{2t}; \end{cases}$$

it clears the way for us to tackle the theorem with gusto.

Proof. Four main cases arise, the easiest of which occur when $b_{4t-1} < d \leq b_{4t}$. This is the region where $h_d = 2t$ is even.

Since $b_{4t-1} = 2t^2 - t$ and $b_{4t} = 2t^2$, we have $d = 2t^2 - q$, for some q with $0 \leq q < t$. Let $p = t$ and $r = t - q$. Then $H(p, q, r) = 2t$ and $N(p, q, r) = 2t^2 - q = d$, QED.

The next-to-easiest cases arise when $2t^2 - 3t + 1 = b_{4t-3} < d \leq b_{4t-2} \leq 2t^2 - 2t + 1$, namely the region where $h_d = 2t - 1$ is odd. We've already ruled out the exceptional situations where $d = b_{4t-2}$. Otherwise $d = 2t^2 - 2t - q$, for some q with $0 \leq q < t - 1$. Hence $H(t, q, t - 1 - q) = 2t - 1$ and $N(t, q, t - 1 - q) = 2t(t - 1) - q = d$, QED.

Good: All cases where h_d is an integer have now been established. Let's try next the region $b_{4t-2} < d \leq b_{4t-1}$, where $h_d = 2t - \frac{1}{2}$. Exactly $b_{4t-1} - b_{4t-2} = t - 1$ values of d need to be covered. Our goal is to find integers p , q , and r such that $H(p, q, r) = \lceil h_d \rceil = 2t$ and $N(p, q, r) = d$. Let's call this the "2t problem."

For example, the 100 problem asks for (p, q, r) with $p+q+r = 100$ and $2p(q+r) - q = d$, given any value d between $b_{198} + 1 = 4902$ and $b_{199} = 4950$. The smallest case, $d = 4902$, turns out to have five solutions: $(p, q, r) = (43, 0, 57), (44, 26, 30), (45, 48, 7), (56, 26, 18), (57, 0, 43)$; and the largest case, $d = 4950$, turns out to have eleven: $(45, 0, 55), (46, 18, 36), (47, 32, 21), (48, 42, 10), (49, 48, 3), (50, 50, 0), (51, 48, 1), (52, 42, 6), (53, 32, 15), (54, 18, 28), (55, 0, 45)$. We'd like to find "structured" and "understandable" solutions that apply to all values of d between these extremes.

A new idea now comes to our rescue. Let $c_k = k^2 - k + 1$, and notice that this sequence $\langle 1, 3, 7, 13, 21, 31, 43, \dots \rangle$ has some beautiful properties [3]; for example, $c_{k+1} = k^2 + k + 1$, hence $c_k c_{k+1} = k^4 + k^2 + 1 = c_{k^2+1}$. Of principal importance to us is the fact that the smallest value of d in the $2t$ problem, $b_{4t-2} + 1 = 2t^2 - 2t + 2$, is equal to $2c_t$.

Say that a triple (p, q, r) is " t -nice" if it solves the $2t$ problem for that smallest value of d . In other words, a t -nice triple satisfies $H(p, q, r) = 2t$ and $N(p, q, r) = 2c_t$. One example is the triple $(c_k, 0, c_{k+1})$, which is $(k^2 + 1)$ -nice according to the formulas above.

The nice thing about a t -nice triple (p, q, r) is that it gives us solutions $(p, q, r), (p, q-1, r+1), \dots, (p, 0, r+q)$ to the $2t$ problem for each of the smallest $q+1$ values of d . Therefore the $2t$ problem will be completely solved if we can find a t -nice triple (p, q, r) with $q \geq t-2$.

Unfortunately, our nice example triple $(c_k, 0, c_{k+1})$ isn't very nice in this respect, because it has $q = 0$. But hooray, we're in luck: If (p, q, r) is t -nice and $r > 0$, it's easy to check that $(p+1, q+2, r-1)$ is $(t+1)$ -nice! Therefore we can generate nice triples that have larger and larger values of q .

Could that be enough to solve every $2t$ problem? Let's take stock of what we've got so far:

(1, 0, 1) is 1-nice	(1, 0, 3) is 2-nice	(3, 0, 7) is 5-nice	(7, 0, 13) is 10-nice	(13, 0, 21) is 17-nice
(2, 2, 0) is 2-nice	(2, 2, 2) is 3-nice	(4, 2, 6) is 6-nice	(8, 2, 12) is 11-nice	(14, 2, 20) is 18-nice
	(3, 4, 1) is 4-nice	(5, 4, 5) is 7-nice	(9, 4, 11) is 12-nice	(15, 4, 19) is 19-nice
	(4, 6, 0) is 5-nice	(6, 6, 4) is 8-nice	(10, 6, 10) is 13-nice	(16, 6, 18) is 20-nice
		(7, 8, 3) is 9-nice	(11, 8, 9) is 14-nice	(17, 8, 17) is 21-nice
		(8, 10, 2) is 10-nice	(12, 10, 8) is 15-nice	(18, 10, 16) is 22-nice
		(9, 12, 1) is 11-nice	(13, 12, 7) is 16-nice	(19, 12, 15) is 23-nice
		(10, 14, 0) is 12-nice	(14, 14, 6) is 17-nice	(20, 14, 14) is 24-nice
			(15, 16, 5) is 18-nice	(21, 16, 13) is 25-nice
			(16, 18, 4) is 19-nice	(22, 18, 12) is 26-nice
			(17, 20, 3) is 20-nice	(23, 20, 11) is 27-nice
			(18, 22, 2) is 21-nice	(24, 22, 10) is 28-nice
			(19, 24, 1) is 22-nice	(25, 24, 9) is 29-nice
			(20, 26, 0) is 23-nice	(26, 26, 8) is 30-nice
			

If q_t denotes the largest value of q for which we've specified a t -nice triple, the sequence begins

$$t = 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \ 18 \ 19 \ 20 \ 21 \ 22 \ 23 \ 24 \ 25$$

$$q_t = 0 \ 2 \ 2 \ 4 \ 6 \ 2 \ 4 \ 6 \ 8 \ 10 \ 12 \ 14 \ 6 \ 8 \ 10 \ 12 \ 14 \ 16 \ 18 \ 20 \ 22 \ 24 \ 26 \ 14 \ 16$$

and in general we have $q_t = 2t - (2k^2 + 2)$ when $2(k-1)^2 + (k-1) + 2 < t \leq 2k^2 + k + 2$. We've solved the $2t$ problem when $q_t \geq t-2$, namely when $t \in [1..5], [8..12], [18..23], [32..38], [50..57], \dots$; in particular, the 100 problem is solved. But infinitely many ever-widening gaps remain.

All is not lost. Let's say that " (p, q, r) does t except for k " if it solves the $2t$ problem when $d = 2c_t + k$. In such a case, the sequence of triples $(p, q, r), (p, q-1, r+1), \dots, (p, 0, r+q)$ will solve the $2t$ problem for $q+1$ consecutive values beginning with $2c_t + k$.

Notice that, by definition, every t -nice triple does t except for 0. And there's wonderful news: If $(t-a, t+a, 0)$ does t except for k , one can check that $(t+1-a, t+1+a, 0)$ does $t+1$ except for $k+1$.

Therefore the fact $(4, 6, 0)$ is 5-nice implies that $(5, 7, 0)$ does 6 except for 1, and that $(6, 8, 0)$ does 7 except for 2. When appended to the nice triples $(4, 2, 6)$ and $(5, 4, 5)$, we're blessed with triples galore that completely solve the $2t$ problem for $t = 6$ and 7.

Similarly, the fact that $(10, 14, 0)$ is 12-nice implies that $(11, 15, 0)$ does 13 except for 1; $(12, 16, 0)$ does 14 except for 2; \dots ; and $(15, 19, 0)$ does 17 except for 5. This completely solves problem $2t$ also for $t = 13, 14, \dots, 17$. And so on, QED.

To complete the proof of the theorem, we're left with the range of values $2t^2 = b_{4t} < d \leq b_{2t+1} = 2t^2 + t$, where $h_d = 2t + \frac{1}{2}$. We shall call it the “ $2t+1$ problem,” because it's quite similar to the $2t$ problem: The remaining goal is to find integers p, q , and r for which $H(p, q, r) = \lceil h_d \rceil = 2t+1$ and $N(p, q, r) = d$.

Following the strategy that worked above, we say that (p, q, r) is $(t + \frac{1}{2})$ -nice if solves the $2t+1$ problem for the smallest value of d , namely $d = 2t^2 + 1$. For example, $(1, 1, 1)$ is 1.5-nice; $(2, 1, 4)$ is 3.5-nice; $(5, 1, 9)$ is 7.5-nice; $(10, 1, 16)$ is 13.5-nice; and in general $((k-1)^2 + 1, 1, k^2)$ is $(c_k + \frac{1}{2})$ -nice for all $k \geq 1(!)$.

Furthermore, when (p, q, r) is $(t + \frac{1}{2})$ -nice, $(p+1, q+2, r-1)$ is $(t + \frac{3}{2})$ -nice.

Aha. By exploiting these new tools, we've got another nice tableau:

(1, 1, 1) is 1.5-nice	(2, 1, 4) is 3.5-nice	(5, 1, 9) is 7.5-nice	(10, 1, 16) is 13.5-nice	(17, 1, 25) is 21.5-nice
(2, 3, 0) is 2.5-nice	(3, 3, 3) is 4.5-nice	(6, 3, 8) is 8.5-nice	(11, 3, 15) is 14.5-nice	(18, 3, 24) is 22.5-nice
	(4, 5, 2) is 5.5-nice	(7, 5, 7) is 9.5-nice	(12, 5, 14) is 15.5-nice	(19, 5, 23) is 23.5-nice
	(5, 7, 1) is 6.5-nice	(8, 7, 6) is 10.5-nice	(13, 7, 13) is 16.5-nice	(20, 7, 22) is 24.5-nice
	(6, 9, 0) is 7.5-nice	(9, 9, 5) is 11.5-nice	(14, 9, 12) is 17.5-nice	(21, 9, 21) is 25.5-nice
.....				

Letting q'_t be the largest q for which we've specified a $(t + \frac{1}{2})$ -nice triple, we now have

$$\begin{aligned}
 t &= 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23\ 24\ 25 \\
 q'_t &= 1\ 3\ 3\ 5\ 7\ 9\ 1\ 3\ 5\ 7\ 9\ 11\ 13\ 15\ 17\ 19\ 9\ 11\ 13\ 15\ 17\ 19\ 21\ 23\ 25
 \end{aligned}$$

and we've solved the $2t+1$ problem when $q'_t \geq t$. In particular, we're done when $t \in [1..6], [13..16], [25..29], [41..46], \dots, [2k^2 - 2k + 1..2k^2 - k + 1]$.

To fill the remaining gaps, let's adapt the trick that worked before: Say that “ (p, q, r) does $t + \frac{1}{2}$ except for k ” if it solves the $2t+1$ problem when $d = 2t^2 + 1 + k$. That yields solutions $(p, q-j, r+j)$ for $d+j$ when $0 \leq j \leq q$. And if $(t-a, t+a+1, 0)$ does $t + \frac{1}{2}$ except for k , it follows that $(t+1-a, t+a+2, 0)$ does $t + \frac{3}{2}$ except for $k+1$.

OK, we're really done! To handle $t = 7, 8, 9, 10, 11, 12$, we use the facts that $(6, 9, 0)$ does 7.5 except for 0; $(7, 10, 0)$ does 8.5 except for 1; $(8, 11, 0)$ does 9.5 except for 2; \dots ; $(11, 14, 0)$ does 12.5 except for 5. And similar facts fill the other gaps.

The theorem is now fully proved. Kinda pretty, eh?

References.

- [1] Donald E. Knuth, *The Art of Computer Programming*, Volume 4, Fascicle 7: *Constraint Satisfaction* (Hoboken, N. J.: Addison-Wesley, 2025), xiv+281 pp. (Soon to be xiv+283 pp.)
- [2] Shisheng Li, “Densities of label- d cells in Fillomino patterns: exact values and constructions for $d \leq 25$,” draft manuscript (18 June 2026).
- [3] OEIS A2061, “Central polygonal numbers.” <https://oeis.org/A002061>; part of the On-Line Encyclopedia of Integer Sequences[®], founded in 1964 by N. J. A. Sloane.