A Curious Sequence of Sets D E Knuth 07 June 07; updated 17 January 08

Consider the sets S_1, S_2, S_3, $\ldots$ defined by the following rules.

Rule 1: $(1,0) \in S_1$.

Rule 2: If $(x,y) \in S_n$ and $(x',y') \in S_{n'}$ then

$$(x,y) \circ (x',y') = \bigl(x+x',\ \min(x+y',x'+y)\bigr) \in S_{n+n'}.$$

Rule 3: If $(x,y) \in S_n$ and $(x',y') \in S_{n'}$ then

$$(x,y) \bullet (x',y') = \bigl(x+x'+\min(y,y'),\ \max(y,y')\bigr) \in S_{n+n'}.$$

Rule 4: There's no other way to become a member of these sets.

Thus we have

$$\begin{aligned} S_1 &= \{(1,0)\},\\ S_2 &= \{(2,0),(2,1)\},\\ S_3 &= \{(3,0),(3,1),(3,2)\},\\ S_4 &= \{(4,0),(4,1),(4,2),(4,3),(5,1)\}, \end{aligned}$$

and so on. Clearly $x > y \geq 0$ for all $(x,y) \in S_n$; also $x \geq n$.

Consider the related sequences of numbers $\langle a_n\rangle$ and $\langle b_n\rangle$ defined by

$$\begin{aligned} a_0 &= 1\,; \\ b_0 &= 0\,; \end{aligned} \qquad \left.\begin{aligned} a_{n+1} &= 2a_n \\ b_{n+1} &= a_n + b_n \end{aligned}\right\}, \text{ if } n \text{ is even;} \qquad \left.\begin{aligned} a_{n+1} &= 2a_n + b_n \\ b_{n+1} &= b_n \end{aligned}\right\}, \text{ if } n \text{ is odd.}$$

Then, by induction, $(a_n,b_n) \in S_{2^n}$, using Rule 2 and Rule 3 alternately. Furthermore we have

$$\begin{aligned} a_n &= 2^{(n-2)/2}F_{n+3}, \quad & b_n &= 2^{(n-2)/2}F_n, \quad && \text{when } n \text{ is even;}\\ a_n &= 2^{(n-1)/2}F_{n+2}, \quad & b_n &= 2^{(n-1)/2}F_{n+1}, \quad && \text{when } n \text{ is odd.} \end{aligned}$$

Thus $a_{n+2}/a_n \approx b_{n+2}/b_n \approx 2\phi^2$ when n is large. In general, the elements of S_{4N} seem to be roughly like the elements of S_N, but multiplied by $2\phi^2$ (which is $3+\sqrt{5} \approx 5.236$).

Let β be the exponent such that $4^\beta = 2\phi^2$. Then $\beta = 1/2 + \lg\phi \approx 1.194242$.

The attached illustrations for small n show how these curious sets begin to develop. Each diagram has been scaled by a factor of $1/n^\beta$. Perhaps the sets $S_{\lfloor 4^n\alpha\rfloor}/(2^n\phi^{2n})$ converge to a limiting set S_α for all real constants $\alpha > 0$. (But the sets $S_{\lfloor 2^n\alpha\rfloor}/(2^{n/2}\phi^n)$ do *not* seem to converge.)

In the original version of these notes (June 2007), I stated the following weak conjecture, which I needed for Section 7.1.4 of *The Art of Computer Programming*: "$(x,y) \in S_n$ implies $x = O(n^\beta)$." And on the basis of empirical results, I also said, "I cannot believe that it could be false." Several people helped me with this problem during the summer, and indeed the conjecture turns out to be true in a strong form:

Theorem (A. X. Chang and V. I. Spitkovsky, October 2007)**.** *If $(x,y) \in S_n$ we have $\sqrt{x^2+y^2} \leq n^\beta$.*

A remarkable thing happens if we change Rule 1 so that the "seed" value is $(1,1)$ instead of $(1,0)$. Rules 2, 3, and 4 stay unchanged, but we call the new sets T_n instead of S_n. Then

$$\begin{aligned} T_1 &= \{(1,1)\},\\ T_2 &= \{(2,2),(3,1)\},\\ T_3 &= \{(3,3),(4,2),(5,1)\},\\ T_4 &= \{(4,4),(5,3),(6,2),(6,4),(7,1)\}, \end{aligned}$$

and in general it turns out that $T_n = \{(x+y,x-y) \mid (x,y) \in S_n\}$. (I let you prove this for yourself.) The funny thing is that Rules 2 and 3 essentially change places: Elements of T_n obtained from Rule 2 are those of S_n obtained from Rule 3, and vice versa!

Let $s_n = \max\{x \mid (x,y) \in S_n\}$ and $t_n = \max\{x \mid (x,y) \in T_n\} = \max\{x+y \mid (x,y) \in S_n\}$. These sequences begin as follows:

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
$s_n =$	1	2	3	5	6	8	9	11	13	14	16	18	19	21	23	26	27	29	31	33	35	37	39	42	43	45	47	49	51	53	55	58
$t_n =$	1	3	5	7	9	11	13	16	18	20	22	26	28	30	32	36	38	42	44	46	49	52	54	58	60	63	67	69	71	75	78	84

The elements of $\langle a_n \rangle$ and $\langle b_n \rangle$ seem to show up here. Namely, $s_1 = a_0 = \frac{1}{2}F_3$; $t_2 = a_1 + b_1 = F_4$; $s_4 = a_2 = F_5$; $t_8 = a_3 + b_3 = 2F_6$; $s_{16} = a_4 = 2F_7$; $t_{32} = a_5 + b_5 = 4F_8$; $s_{64} = a_6 = 4F_9$; and $t_{128} = a_7 + b_7 = 8F_{10}$. It is natural to conjecture that this pattern will continue, namely that $s_{2^n} = a_n$ when n is even, $t_{2^n} = a_n + b_n$ when n is odd. (This new conjecture has been verified for s_{2^n} and t_{2^n} when $n \le 15$.)

Relation to prior work.

An important paper by Martin Sauerhoff, Ingo Wegener, and Ralph Werchner [*Discrete Applied Mathematics* **103** (2000), 237–258] explains the motivation for this whole business. It also claims to have proved the weak conjecture stated earlier. But unfortunately the complex argument given there is flawed.

If you want to read that paper, let me warn you that their notation is quite different from mine, and they use a more complicated recurrence. Basically, they define pairs of values "$\bigl(\text{size}(f), \text{size}(f, \overline{f})\bigr)$," which correspond to what I have called $(x+y, 2x)$ when $(x,y) \in S_n$. They define a function $\varphi(s,t)$ and claim in Lemma 12 that $\varphi(x+y, 2x) \le \varphi(1,2)n^\beta$. This would prove the weak conjecture, except that Lemma 12 depends on Lemma 11, which is incorrect.

In their definition of φ they give intricate formulas for quantities called p_1, p_2, q_1, q_2; in fact, simpler formulas are available, namely, $p_1 = \phi/\sqrt{2}$, $p_2 = \phi^2/\sqrt{2}$, $q_1 = \phi^2/2$, and $q_2 = \phi$. Then they give numeric values for five constants obtained by solving a set of simultaneous linear equations. Actually these constants are $a_1 = \sqrt{2} + 1 - \sqrt{5}$, $a_2 = a_1/(\sqrt{2} - 1)$, $a_1' = 1 + \sqrt{5} - 2\sqrt{2}$, $a_2' = a_1'/\sqrt{2}$, and $a_1'' = 3 - \sqrt{5}$. We have $a_1'/a_1 = \sqrt{2}\phi$ and $a_2'/a_2 = (\sqrt{2} - 1)\phi$. In their Fig. 2, the line between regions I and II is "$t = \phi s$" and the line between II and III is "$t = (2/\phi)s$"; this constant $2/\phi$ can also be written $1 + \phi^{-3}$.

To see why their proof of Lemma 11 fails, consider (in their notation) the case $s = \phi$, $t = \phi^2$, $m = \varphi(s,t)^{1/\beta} \approx 1.3367$, $\tilde{s} = 1$, $\tilde{t} = 2$, $\widetilde{m}(\tilde{s}, \tilde{t}) = \varphi(1,2) \approx 1.032$. They claim that $\varphi(s + \tilde{t}, t + \tilde{t}) \le (m + \widetilde{m})^\beta$. But $\varphi(s + \tilde{t}, t + \tilde{t}) \approx 2.806$ and $(m + \widetilde{m})^\beta \approx 2.801$.

Even if the argument could be patched, it is messy and unsatisfying— certainly nothing I would like to put into *The Art of Computer Programming*. Wegener himself, in his monograph on *Branching Programs and Binary Decision Diagrams* (page 23), says simply, "The tedious and mathematically involved analysis is described [in the publication] and leads to the stated result."

Chang and Spitkovsky's new proof, of a stronger and cleaner result, is quite interesting and certainly not tedious. But still, this is a theorem about a sequence of integers, and the methods of Chang and Spitkovsky are highly analytical; they give no hint about how to prove the conjectured relations between the integer sequences $\langle a_n \rangle$, $\langle b_n \rangle$, $\langle s_n \rangle$, and $\langle t_n \rangle$. Isn't there some elementary way to understand what is going on here?

A plausible line of attack?

If I had time to study further, I think I would try something like this, based on the idea that Rules 2 and 3 essentially define binary operations on the pairs (x,y) and (x',y'). Those binary operations are associative and commutative, so I would extend them to k-ary operations.

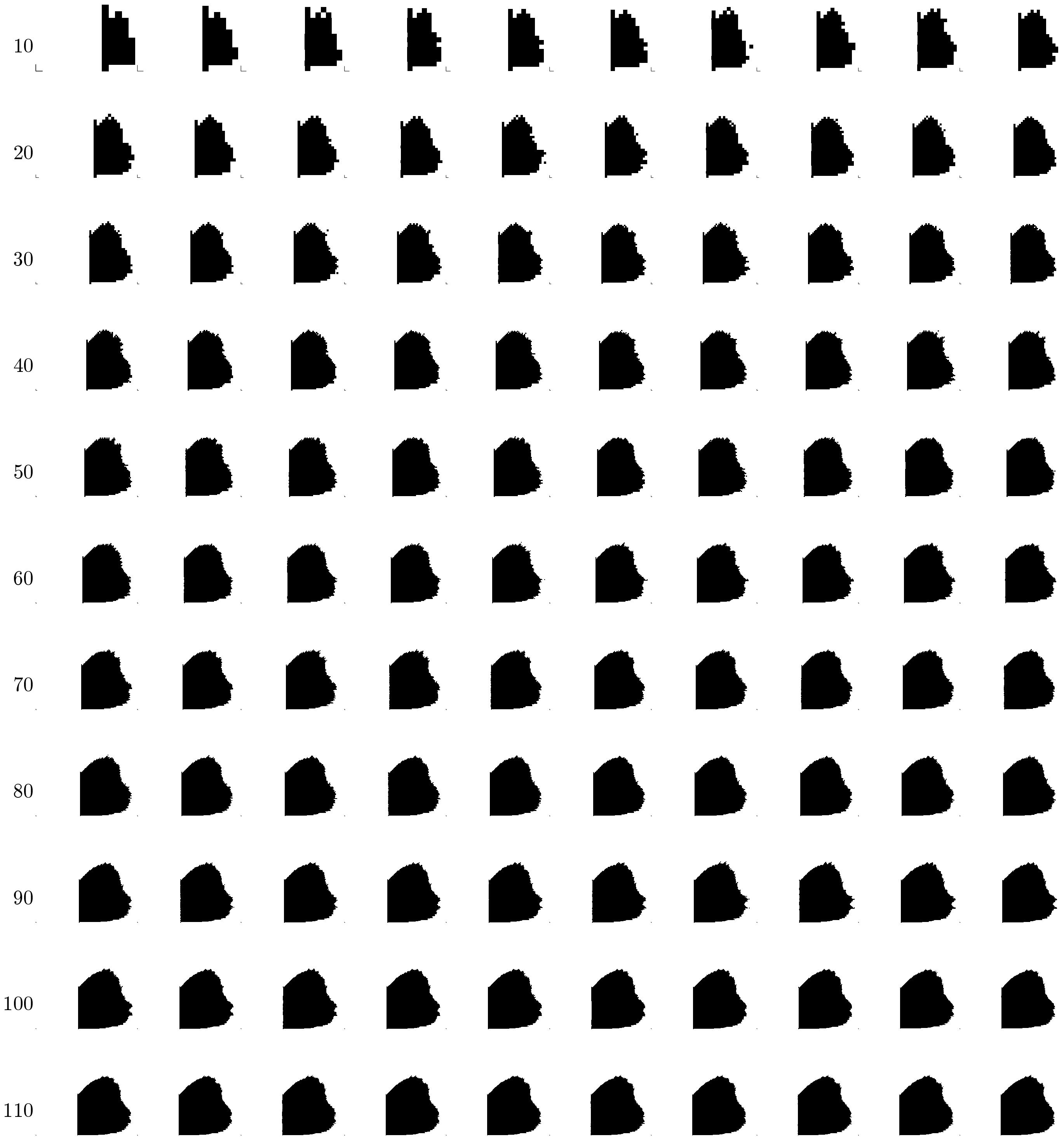
Suppose $(x_1, y_1) \in S_{n_1}$, $(x_2, y_2) \in S_{n_2}$, …, $(x_k, y_k) \in S_{n_k}$. Then the k-ary Rule 2 states that

$$\bigl(x_1 + x_2 + \cdots + x_k,\ x_1 + x_2 + \cdots + x_k + \min(y_1 - x_1, y_2 - x_2, \ldots, y_k - x_k)\bigr) \in S_{n_1 + n_2 + \cdots + n_k};$$

and the k-ary Rule 3 states that

$$\bigl(x_1 + x_2 + \cdots + x_k + y_1 + y_2 + \cdots + y_k - \max(y_1, y_2, \ldots, y_k),\ \max(y_1, y_2, \ldots, y_k)\bigr) \in S_{n_1 + n_2 + \cdots + n_k}.$$

Now let $U_1 = V_1 = \{(1,0)\}$. For $n > 1$, let U_n be all the pairs you can get by applying the k-ary Rule 2 to the sets V_1, …, V_{n-1}; and let V_n be all the pairs you can get by applying the k-ary Rule 3 to the sets U_1, …, U_{n-1}. Then I think $S_n = U_n \cup V_n$. Further exploration by members of the "curious research group" have revealed fascinating patterns (for example, when n is a multiple of 3), which are still being investigated.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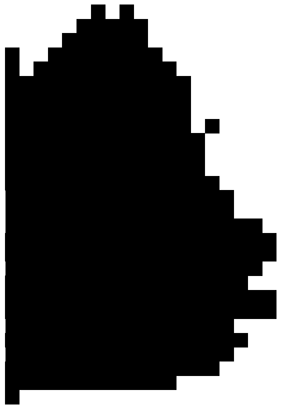
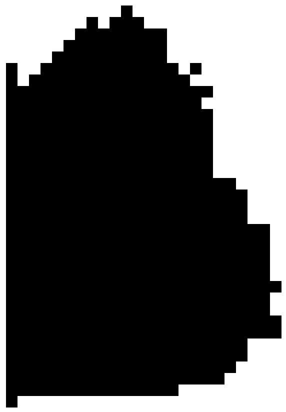
$n = 25$



$n = 100$

$n = 30$



$n = 120$