Midterm Exam

Good luck to all.
—Friendly TA

THIS IS A TAKE-HOME-and-open-everything-but-don't-get-help-from-other-people exam, due in class Wednesday, November 8. There is no time limit. USE FIVE BLUE BOOKS, one for each of the five problems, SHOWING ALL YOUR WORK (so that partial credit can be given for incomplete answers). PLEASE SIGN YOUR NAME ON THE COVER OF EACH BLUE BOOK.

Problem 1: A double recurrence. (20 points)

Find and prove a simple closed form for the numbers $\int{n \atop k}\int$ defined as follows for all integers n and k:

$$\begin{aligned} \int{n \atop k}\int &= (-1)^k \int{n-1 \atop k}\int + \int{n-1 \atop k-1}\int \,; \\ \int{n \atop 0}\int &= 1\,; \\ \int{0 \atop k}\int &= [k=0]\,. \end{aligned}$$

(Be sure to verify your formula when n and k are negative as well as positive.)

Problem 2: A harmonic congruence. (20 points)

a Suppose H_n is written as a fraction a_n/b_n in lowest terms. What is the highest power of 2 that divides b_n?

b The "exclusive or" of two nonnegative integers m and n, written $m \oplus n$, is obtained by adding m and n in binary notation and ignoring all carry bits. For example,

$$19 \oplus 25 = (10011)_2 \oplus (11001)_2 = (01010)_2 = 10\,.$$

Prove that, if $0 \leqslant m < n$ and if the fraction $\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}$ is reduced to lowest terms, then the highest power of 2 dividing the denominator is the same as the highest power of 2 dividing the denominator of $H_{m \oplus n}$.

c Find all integer solutions $m, n \geqslant 0$ to the congruence $H_m \equiv H_n \pmod 1$.

Problem 3: A golden logarithm. (10 points)

Prove that $\sum_{n \geqslant 1} F_n/(n2^n) = C \ln \phi$, where C^2 is a rational number, and determine the value of C.

Problem 4: A textbook summation. (25 points)

Keep your cool while watching these floors and ceilings crumble.

Let p be a prime number and let m, n be positive integers. Find a closed form for the following expression:

$$\sum_{0\leqslant j<p}\left(\left\lfloor\frac{\displaystyle\sum_{1\leqslant k\leqslant mj \bmod p}\binom{k}{\lfloor\ln(m+n)\rfloor}H_k}{2n+1}\right\rfloor + \left\lceil\frac{2n-\dbinom{m^2j \bmod p}{\lceil\ln(m+n)\rceil}\left(H_{m^2j \bmod p}-\dfrac{1}{\lceil\ln(m+n)\rceil}\right)}{2n+1}\right\rceil\right).$$

Problem 5: A concrete aftermath. (25 points)

Two building inspectors are examining n buildings arranged in a circle. The first inspector, who is looking for asbestos, examines the buildings in the natural order $1, 2, \ldots, n$. But the second inspector, who is looking for structural damage, examines them in "Josephus order" $2, 4, \ldots, J(n)$, always skipping past one yet-unchecked building before going into another. Both types of inspections take the same amount of time.

Log n times around the quad.

For example, when $n = 5$ the order of building inspections is

First inspector	1	2	3	4	5
Second inspector	2	4	1	5	3.

When $n = 10$ it is

First inspector	1	2	3	4	5	6	7	8	9	10
Second inspector	2	4	6	8	10	3	7	1	9	5.

Notice that in the latter case the inspectors bump into each other, when they are examining buildings 7 and 9; this does not happen when $n = 5$.

a Prove that if $n \bmod 3 = 1$, there is always a time when the inspectors both work in the same building. *Hint:* Consider assigning new numbers to buildings that are skipped over, as in Section 3.3 of the text.

b Prove that if $n \bmod 7 = 3$, there is always a time when the two inspectors both work in the same building.

c Prove that, in fact, the inspectors share a building at some time if and only if $n \bmod (2^p - 1) = 2^{p-1} - 1$ for at least one prime number p.