Final Exam

THIS IS A TAKE-HOME-and-open-everything-but-don't-get-help-from-other-people exam, due in room MJH 326 on Wednesday, December 13 before 5 pm. There is no time limit. USE FOUR BLUE BOOKS, one for each of the four problems, SHOWING ALL YOUR WORK (so that partial credit can be given for incomplete answers). PLEASE SIGN YOUR NAME ON THE COVER OF EACH BLUE BOOK.

The problems have been designed so that you can almost always work each part independently, without having solved the previous parts. Therefore, don't give up on a problem just because you're stumped on part (a).

There are many students in this course, yet all exams need to be graded in 43 hours. PLEASE help by making your exam easy to grade. Think through your presentation before writing in the blue books. If you work out examples, just tabulate the results; don't give all the gory details. If you have significant intermediate results, label them for easy reference. Reserve your "stream of consciousness" abilities for another opportunity. However, don't be too sketchy; you need to convince the reader that you know what you are doing. Don't assume that the reader is clairvoyant; explain all steps. If you are proving something by contradiction, make your temporary assumptions stand out clearly.
—Friendly TAs

Problem 1: Some sums. (20 points)

a Let $\sum_{\pi(n)}$ denote a sum over all permutations $\pi_1\pi_2\ldots\pi_n$ of $\{1,2,\ldots,n\}$. Prove that if $x_1, x_2, \ldots, x_n$ are nonzero, the sum

$$\sum_{\pi(n)} \frac{1}{x_{\pi_1}(x_{\pi_1}+x_{\pi_2})\ldots(x_{\pi_1}+\cdots+x_{\pi_n})} = \sum_{\pi(n)} \prod_{k=1}^{n} \Bigl(\sum_{j=1}^{k} x_{\pi_j}\Bigr)^{-1}$$

can be written in a very simple form. For example, when $n=3$ the sum to be simplified is

$$\frac{1}{x_1(x_1+x_2)(x_1+x_2+x_3)} + \frac{1}{x_1(x_1+x_3)(x_1+x_3+x_2)} + \frac{1}{x_2(x_2+x_1)(x_2+x_1+x_3)} + \frac{1}{x_2(x_2+x_3)(x_2+x_3+x_1)} + \frac{1}{x_3(x_3+x_1)(x_3+x_1+x_2)} + \frac{1}{x_3(x_3+x_2)(x_3+x_2+x_1)}\,.$$

b Use part (a) to prove that

$$\sum_{k_1,k_2,\ldots,k_n\in K} \frac{z^{k_1+k_2+\cdots+k_n}}{x_{k_1}(x_{k_1}+x_{k_2})\ldots(x_{k_1}+\cdots+x_{k_n})} = \frac{1}{n!}\Bigl(\sum_{k\in K}\frac{z^k}{x_k}\Bigr)^n,$$

if K is any set of integers such that $x_k \neq 0$ for all $k \in K$.

c Now derive the amazing, almost incredible identity

$$\sum_{\substack{k_1+\cdots+k_n=m\\ k_1,\ldots,k_n\geqslant 0}} \frac{1}{k_1!\,(k_1!+k_2!)\ldots(k_1!+\cdots+k_n!)} = \frac{n^m}{m!\,n!}\,.$$

For example, when $m=2$ and $n=3$ this sum is

$$\begin{aligned}\frac{1}{0!(0!+0!)(0!+0!+2!)} &+ \frac{1}{0!(0!+1!)(0!+1!+1!)}\\ + \frac{1}{0!(0!+2!)(0!+2!+0!)} &+ \frac{1}{1!(1!+0!)(1!+0!+1!)}\\ + \frac{1}{1!(1!+1!)(1!+1!+0!)} &+ \frac{1}{2!(2!+0!)(2!+0!+0!)} = \frac{9}{12}\,.\end{aligned}$$

d Find a closed form for the similar sum

$$\sum_{\substack{k_1+\cdots+k_n=m\\ k_1,\ldots,k_n\geqslant 1}} \frac{1}{k_1!\,(k_1!+k_2!)\ldots(k_1!+\cdots+k_n!)}\,.$$

Problem 2: Friendly flips. (30 points)

Alice and Bill are at it again, flipping coins. But this time they're playing another game: At each step, each of them flips a (fair) coin; and if the total number of heads thrown so far by Alice equals the total number of tails thrown so far by Bill, they shake hands and smile at each other.

a Let P_m be the probability that Alice and Bill shake hands after their mth toss. Prove that $P_m = (-1)^m\binom{-1/2}{m}$.

b What is the average number of handshakes, if they each flip n times? Give your answer in closed form.

c Find the asymptotic value of this average number, with absolute error $O(n^{-1})$. *Hint:* Exercise 5.22 may come in handy.

It's OK to use the result of an exercise without doing the exercise.

d Let $P_{l,m}$ be the probability that Alice and Bill shake hands after their lth toss *and* after their mth toss, when $l<m$. Find a closed form for this quantity $P_{l,m}$. *Hint:* This is trivial.

e Let the random variable X be the total number of handshakes when Alice and Bill each make n tosses. Prove that the expected value of X^2 is

$$\sum_{1\leqslant m\leqslant n} P_m + 2\sum_{1\leqslant l<m\leqslant n} P_{l,m}\,.$$

f Show that in this game we have $E(X^2) = 2n - 3(EX)$.

g What is the standard deviation of the number of handshakes after n flips? Give your answer as an asymptotic formula correct to $O(1)$.

h Suppose Alice and Bill each play with a biased coin that comes up heads with probability p. Prove that if $p \neq \frac{1}{2}$, the average number of handshakes they make after n flips approaches a finite limit as $n \to \infty$, and evaluate that limit in closed form.

Problem 3: Inflated statistics. (25 points)

The newspapers tell us that the recent earthquake was the most expensive natural disaster in history. However, this world record is not quite so impressive when we realize that it doesn't take inflation into account.

The purpose of Problem 3 is to study the expected number of world records that occur in a sequence of n natural disasters, when effects of inflation are included. We assume for simplicity that all monetary amounts increase by a factor of $1+\epsilon$ between disasters.

Suppose $x_0, x_1, \ldots, x_{n-1}$ are real numbers such that x_k is chosen uniformly at random between 0 and $(1+\epsilon)^k$, independent of the other x's, where $\epsilon \geqslant 0$. We want to study the number of record-breaking x's, namely

When $k=0$, the statement $x_k > x_0, \ldots, x_{k-1}$ is always true.

$$M_n(\epsilon) \;=\; E\biggl(\sum_{k=0}^{n-1}[x_k > x_0, \ldots, x_{k-1}]\biggr) \;=\; \sum_{k=0}^{n-1}\Pr(x_k > x_0, \ldots, x_{k-1}).$$

In this problem we have *continuous* probabilities, not discrete ones, so the probability of an event is calculated by integration instead of by summation:

$$\Pr\bigl((x_0, \ldots, x_{n-1}) \in A\bigr) \;=\; \frac{\int_0^1 dx_0 \int_0^{1+\epsilon} dx_1 \ldots \int_0^{(1+\epsilon)^{n-1}} dx_{n-1}\,\bigl[(x_0, \ldots, x_{n-1}) \in A\bigr]}{\int_0^1 dx_0 \int_0^{1+\epsilon} dx_1 \ldots \int_0^{(1+\epsilon)^{n-1}} dx_{n-1}}.$$

Here the notation for integrals is $\int dx\, f(x)$ instead of the usual $\int f(x)\,dx$, because it works out better in this particular problem.

Notice that the denominator is simply $(1+\epsilon)^{\binom{n}{2}}$. For example, the probability that $x_0 > x_1 > x_2$ is

$$\frac{\int_0^1 dx_0 \int_0^{x_0} dx_1 \int_0^{x_1} dx_2}{\int_0^1 dx_0 \int_0^{1+\epsilon} dx_1 \int_0^{(1+\epsilon)^2} dx_2} \;=\; \frac{\int_0^1 dx_0 \int_0^{x_0} dx_1\, x_1}{(1+\epsilon)^3} \;=\; \frac{\int_0^1 dx_0\, x_0^2/2}{(1+\epsilon)^3} \;=\; \frac{1}{6(1+\epsilon)^3}.$$

a Show that if $j < k$ we have

$$\Pr(x_j \geqslant x_k > x_{j+1}, \ldots, x_{k-1}) \;=\; \frac{1}{(k-j+1)(k-j)(1+\epsilon)^{\binom{k-j+1}{2}}}.$$

(This is the probability that x_k is *not* a record and that x_j was the most recent disaster costing at least as much.)

b Now give an asymptotic expression for $\Pr(x_k > x_0, \ldots, x_{k-1})$, having absolute error $O(\epsilon^2)$ as $\epsilon \to 0$, assuming that k is constant.

c Sum on k to get $M_n(\epsilon)$ with absolute error $O(\epsilon^2)$ as $\epsilon \to 0$, when n is constant.

Problem 4: A superpowerful final recurrence. (25 points)

a Let $e \uparrow\uparrow x = e^x$ when $0 \leqslant x < 1$ and $e \uparrow\uparrow x = e^{e \uparrow\uparrow (x-1)}$ when $x \geqslant 1$. Prove that if we are given any monotonic sequence $S_0 < S_1 < S_2 < \cdots$ with the property that

$$\lim_{n\to\infty} S_n = \infty \qquad \text{and} \qquad \ln S_n = S_{n-1} + O(1),$$

then $S_n = e \uparrow\uparrow \bigl(n + O(1)\bigr)$.

b Consider the sequence $\langle A_n \rangle$ defined by the recurrence

$$A_0 = 2; \quad A_1 = 4; \quad A_n = \binom{A_{n-1}}{A_{n-2}} \quad \text{for } n > 1.$$

Prove that $A_n = e \uparrow\uparrow \bigl(n/2 + O(1)\bigr)$. *Hints:* Show that, when n is large, $\ln A_n$ is approximately equal to $A_{n-2} \ln A_{n-1}$. Then prove that $\lim_{n\to\infty} (\ln A_n) / \prod_{k=0}^{n-2} A_k$ exists.