# Exam answers

HERE ARE sketches of solutions to the midterm and final exam problems.

*('M1' means midterm problem 1. Get it?)*

**M1** First we note that the recurrence has at most one solution: The addition formula defines $\int{n \atop k-1}\int$ for all $n$ when $\int{n \atop k}\int$ is defined for all $n$, and it defines $\int{n \atop k}\int$ for all $n$ when $\int{0 \atop k}\int$ is given and $\int{n \atop k-1}\int$ is defined for all $n$. Therefore if $f(n,k)$ is any function that satisfies the given recurrence equations, we must have $\int{n \atop k}\int = f(n,k)$ for all $n$ and $k$.

Experimentation with small cases leads us to conjecture that

$$\int{n \atop k}\int \;=\; \binom{\lfloor n/2\rfloor}{\lfloor k/2\rfloor}[n \text{ odd or } k \text{ even}]\,.$$

Let $f(n,k)$ be the right-hand side of this equation. Setting $n=0$, we have

$$f(0,k) \;=\; \binom{\lfloor 0/2\rfloor}{\lfloor k/2\rfloor}[0 \text{ odd or } k \text{ even}] \;=\; \binom{0}{\lfloor k/2\rfloor}[k \text{ even}] \;=\; [k=0]\,.$$

Setting $k=0$ clearly gives $f(n,0)=1$. So our proof will be complete if we can show that $f(n,k)$ satisfies the addition formula. If $n$ is odd then

$$\begin{aligned}
&(-1)^k f(n-1,k) + f(n-1,k-1)\\
&\qquad = \;(-1)^k\binom{\lfloor n/2\rfloor}{\lfloor k/2\rfloor}[k \text{ even}] + \binom{\lfloor n/2\rfloor}{\lfloor (k-1)/2\rfloor}[k \text{ odd}]\\
&\qquad = \;\binom{\lfloor n/2\rfloor}{\lfloor k/2\rfloor} \;=\; f(n,k)\,.
\end{aligned}$$

And if $n$ is even we have

$$\begin{aligned}
&(-1)^k f(n-1,k) + f(n-1,k-1)\\
&\qquad = \;(-1)^k\binom{\lfloor n/2\rfloor - 1}{\lfloor k/2\rfloor} + \binom{\lfloor n/2\rfloor - 1}{\lfloor (k-1)/2\rfloor}\\
&\qquad = \;\binom{\lfloor n/2\rfloor}{\lfloor k/2\rfloor}[k \text{ even}] \;=\; f(n,k)\,.
\end{aligned}$$

**M2** (a) (Exercise 6.21 is similar.) Consider any sum $a/b + a'/b' = a''/b''$ where all three fractions are in lowest terms, and suppose that $2^k\backslash\backslash b$, $2^{k'}\backslash\backslash b'$,

$2^{k''}\backslash\backslash b''$. Then it's not difficult to prove that $k'' = \max(k, k')$ if $k \neq k'$, while $k'' < k$ if $k = k'$.

The answer is $\lfloor \lg n \rfloor$, since $1/2^{\lfloor \lg n \rfloor}$ is the unique fraction in the sum $\sum_{k=1}^{n} 1/k = H_n$ whose denominator is divisible by this many 2s.

(b) In binary notation we have $m = (\alpha 0 \beta)_2$ and $n = (\alpha 1 \gamma)_2$ for some bit-strings $\alpha$, $\beta$, $\gamma$, where $\beta$ and $\gamma$ have the same length. Hence the highest power of 2 that occurs in the fractions $\frac{1}{m+1} + \cdots + \frac{1}{n}$ occurs uniquely in the term $\frac{1}{k}$ where $k = (\alpha 10 \ldots 0)_2$. This power is $2^{\lfloor \lg (m \oplus n) \rfloor}$.

(c) Clearly $m = n$ is a solution for all $n \geqslant 0$. Otherwise we may assume that $m < n$. Then $\frac{1}{m+1} + \cdots + \frac{1}{n} \equiv 0 \pmod 1$, so the largest power of 2 dividing the denominator of the sum $\frac{1}{m+1} + \cdots + \frac{1}{n}$ must be $2^0$. Hence $\lfloor \lg(m \oplus n) \rfloor = 0$; hence $m \oplus n = 1$, and we must have $n = m + 1$, an odd number. But $\frac{1}{n} \equiv 0 \pmod 1$ iff $n = 1$. So the only solutions with $m \neq n$ are $\{m, n\} = \{0, 1\}$.

**M3** We have $F_n = (\phi^n - \hat{\phi}^n)/\sqrt{5}$ and $\sum_{n \geqslant 1} z^n/n = -\ln(1 - z)$; it follows that the stated sum is $1/\sqrt{5}$ times

$$\ln(1 - \hat{\phi}/2) - \ln(1 - \phi/2) \;=\; \ln \frac{2 - \hat{\phi}}{2 - \phi} \;=\; 4 \ln \phi\,,$$

since $2 - \hat{\phi} = 2 + \phi^{-1} = \phi^2$ and $2 - \phi = -\phi^{-2}$. Thus $C = 4/\sqrt{5}$.

**M4** This problem takes us on a guided tour of the book; at each step there's only one "obvious" thing to do. First, if $p \backslash m$ the sum reduces to zero since $mj \bmod p = m^2 j \bmod p = 0$ for all $j$. Second, if $p \nmid m$ we can split the sum into two parts and then replace both $mj \bmod p$ and $m^2 j \bmod p$ by $j$, because these quantities run through the values $0 \leqslant j < p$ in some order. Third, we can evaluate the sum

$$\sum_{1 \leqslant k \leqslant j} \binom{k}{\lfloor \nu \rfloor} H_k \;=\; \binom{j+1}{\lceil \nu \rceil} \left( H_{j+1} - \frac{1}{\lceil \nu \rceil} \right), \qquad \nu = \ln(m + n)$$

using summation by parts and/or (6.70), since $H_0 = 0$ and $\nu$ is not an integer. Fourth, we can negate the upper index of $\binom{2n-x}{2n+1} = -\binom{x}{2n+1}$ and then use identity (3.4) to change ceiling to floor; the given sum has reduced to a telescoping series

$$\sum_{0 \leqslant j < p} \left( \left\lfloor \binom{\binom{j+1}{\mu}\left(H_{j+1} - \frac{1}{\mu}\right)}{2n+1} \right\rfloor - \left\lfloor \binom{\binom{j}{\mu}\left(H_j - \frac{1}{\mu}\right)}{2n+1} \right\rfloor \right)$$
$$= \left\lfloor \binom{\binom{p}{\mu}\left(H_p - \frac{1}{\mu}\right)}{2n+1} \right\rfloor, \qquad \mu = \lceil \ln(m + n) \rceil.$$

Multiply this last result by $[p \nmid m]$ to get the general answer.

**M5** Assign new numbers as in 3.3. At time $t$, the second inspector is examining the building whose original number is the first element in the sequence

$$2t \bmod (2n+1), \quad 2^2 t \bmod (2n+1), \quad 2^3 t \bmod (2n+1), \quad \ldots$$

that is $\leqslant n$. The other numbers in this sequence are the new numbers assigned to that building.

(a) When $n = 3m+1$ and $t = 2m+1$, this sequence begins with $4m+2$, then comes $(8m+4) \bmod (6m+3) = 2m+1$; the inspectors collide in building $2m+1$. (b) When $n = 7m+3$ and $t = 6m+3$, the sequence begins

$$\begin{aligned} 12m+6, \quad (24m+12) \bmod (14m+7) &= 10m+5, \\ (20m+10) \bmod (14m+7) &= 6m+3, \end{aligned}$$

hence they collide in building $6m+3$. (Notice that both cases (a) and (b) occur when $n = 10$.)

(c) In general suppose that $t$ is the first element $\leqslant n$ in the stated sequence, and suppose $t$ is the $k$th element. Then $k \geqslant 2$, and the sequence must have the form

$$2n+1-u, \quad 2n+1-2u, \quad 2n+1-4u, \quad \ldots, \quad 2n+1-2^{k-1}u$$

where $2t = 2n+1-u$ and $2n+1-2^{k-1}u = t$. These equations imply that $u = (2n+1)/(2^k-1)$ and $t = (2^{k-1}-1)u$. Hence there is at most one such sequence, and it exists iff $2n+1$ is divisible by $2^k-1$. (This argument proves that the number of times the inspectors meet is exactly the number of divisors of $2n+1$ that have the form $2^k-1$.)

Notice that if $2n+1$ is divisible by $2^k-1$ and if $p$ is a prime factor of $k$, then $2n+1$ is divisible by $2^p-1$. So it suffices to test cases when $k$ is prime. Finally, $2n+1$ is divisible by $2^p-1$ iff $2n \equiv -1 \pmod{2^p-1}$ iff $n \equiv 2^{p-1}-1$ $(\mathrm{mod}\ 2^p-1)$.

**F1** (a) By induction it's $1/x_1x_2\ldots x_n$, since this induction hypothesis tells us that the sub-sum when $\pi_n = k$ is

$$\frac{x_k}{x_1x_2\ldots x_n}\,\frac{1}{(x_1+\cdots+x_n)}\,.$$

(b) $\bigl(\sum_{k\in K} z_1^k/x_k\bigr)\ldots\bigl(\sum_{k\in K} z_n^k/x_k\bigr) = \sum_{k_1,\ldots,k_n\in K} z_1^{k_1}\ldots z_n^{k_n}/x_{k_1}\ldots x_{k_n}$, which we know from (a) is

$$\sum_{k_1,\ldots,k_n\in K}\ \sum_{\pi(n)} \frac{z_1^{k_1}\ldots z_n^{k_n}}{x_{k_{\pi_1}}(x_{k_{\pi_1}}+x_{k_{\pi_2}})\ldots(x_{k_{\pi_1}}+\cdots+x_{k_{\pi_n}})}$$

$$= \sum_{k_1,\ldots,k_n\in K}\sum_{\pi(n)} \frac{z_{\pi_1}^{k_{\pi_1}}\ldots z_{\pi_n}^{k_{\pi_n}}}{x_{k_1}(x_{k_1}+x_{k_2})\ldots(x_{k_1}+\cdots+x_{k_n})}\,.$$

Now set $z_1=\cdots=z_n=z$ and divide by $n!$.

(c) Let $K=\{0,1,2,\ldots\}$ and $x_k=k!$; equate coefficients of $z^m$ in the identity of part (b). (d) ${m \brace n}/m!$, by (7.49).

**F2** (a) Alice has flipped exactly $k$ heads with probability $\binom{m}{k}2^{-m}$, and Bill has flipped exactly $k$ tails with probability $\binom{m}{m-k}2^{-m}$. So

*Courtship rituals observed in this tribe of mathematicians were most peculiar.*

$$P_m \;=\; 2^{-2m}\sum_{k=0}^{m}\binom{m}{k}\binom{m}{m-k} \;=\; 2^{-2m}\binom{2m}{m},$$

which equals $(-1)^m\binom{-1/2}{m}$ by (5.37).

(b) According to (5.114) we have

$$\sum_{k<n+1}\binom{-1/2}{k}(-1)^k \;=\; (-1)^n\binom{-3/2}{n},$$

which can also be written $\binom{n+1/2}{n}$. Subtract 1 because the term for $k=0$ should not be included.

(c) One way is to write $\binom{n+1/2}{n}=(2n+1)\binom{2n}{n}/2^{2n}$ and apply Stirling's approximation to this formula. A slightly more difficult, but possibly more instructive, way is to proceed directly as follows: We have

$$\begin{aligned}\ln(n+\tfrac12)! \;&=\; (n+1)\ln(n+\tfrac12)-(n+\tfrac12)+\sigma+\tfrac{1}{12}n^{-1}+O(n^{-2})\,;\\ \ln n! \;&=\; (n+\tfrac12)\ln n-n+\sigma+\tfrac{1}{12}n^{-1}+O(n^{-2})\,;\\ \ln(n+\tfrac12) \;&=\; \ln n+\tfrac12 n^{-1}-\tfrac18 n^{-2}+O(n^{-3})\,.\end{aligned}$$

Hence $\ln(n+\frac12)!-\ln n! = \frac12\ln n+\frac38 n^{-1}+O(n^{-2})$. Taking the exponential of both sides yields

$$\frac{(n+\frac12)!}{n!} \;=\; n^{1/2}+\frac38 n^{-1/2}+O(n^{-3/2})\,,$$

a result that can also be obtained by using exercise 9.44 since $\left[{1/2 \atop -1/2}\right]=\binom{1/2}{2}=-\frac18$. Now divide by $\frac12!$ (see exercise 5.22), and get the answer:

$$2\sqrt{n/\pi}-1+3/(4\sqrt{n\pi})+O(n^{-3/2})\,.$$

(d) $P_{l,m}=P_lP_{m-l}$.

(e) Consider the random variable $X_m=$ [they shake after $m$th flip]. Now $X=X_1+\cdots+X_n$, and we can proceed as in the derivation of (8.24).

(f) We have

$$\begin{aligned}\sum_{1\leqslant l<m} P_{l,m} &= \sum_{1\leqslant l<m}(-1)^l\binom{-1/2}{l}(-1)^{m-l}\binom{-1/2}{m-l}\\ &= \sum_l(-1)^m\binom{-1/2}{l}\binom{-1/2}{m-l}-2(-1)^m\binom{-1/2}{m}\\ &= 1-2P_m\end{aligned}$$

(see the derivation of (5.39)), hence $E(X^2)=\sum_{m=1}^n(2-3P_m)$.

(g) The variance is $E(X^2)-(EX)^2=2n-\binom{n+1/2}{n}^2-\binom{n+1/2}{n}+2$, so the standard deviation is $\sqrt{(2-4/\pi)n}+O(1)$.

(h) In this case $P_m=\binom{-1/2}{m}(-4pq)^m$, where $q=1-p$, so the average number of handshakes approaches $\sum_{m\geqslant1}P_m=(1-4pq)^{-1/2}-1$ (by the binomial theorem).

*Let's resist the temptation to comment about disasters.*

**F3** (a) We can assume that $j=0$ and $k=n-1$, because the variables $x_j\ldots,x_k$ form a sequence of $k-j+1$ consecutive natural disasters. Then the numerator of $\Pr(x_0\geqslant x_{n-1}>x_1,\ldots,x_{n-2})$ is

$$\begin{aligned}&\int_0^1 dx_0\int_0^{x_0}dx_{n-1}\int_0^{x_{n-1}}dx_1\ldots\int_0^{x_{n-1}}dx_{n-2}\\ &\quad=\int_0^1 dx_0\int_0^{x_0}dx_{n-1}\,x_{n-1}^{n-2}\;=\;\int_0^1 dx_0\,\frac{x_0^{n-1}}{n-1}\;=\;\frac{1}{n(n-1)}\,.\end{aligned}$$

(b) Let $\rho=1/(1+\epsilon)$. The probability $P_k$ that $x_k$ is a world record equals 1 minus the probability that it isn't a world record, namely

$$1-\sum_{j=0}^{k-1}\frac{\rho^{\binom{k-j+1}{2}}}{(k-j+1)(k-j)}\;=\;1-\sum_{j=1}^{k}\frac{\rho^{\binom{j+1}{2}}}{j(j+1)}\,;$$

and $\rho^{\binom{j+1}{2}}=(1+\epsilon)^{-\binom{j+1}{2}}=1-\binom{j+1}{2}\epsilon+O(\epsilon^2)$. Thus, for fixed $k$,

$$P_k\;=\;1-\sum_{j=1}^{k}\left(\frac{1}{j(j+1)}-\frac12\epsilon+O(\epsilon^2)\right)\;=\;\frac1k+\frac k2\epsilon+O(\epsilon^2)\,.$$

(c) And $M_n(\epsilon)=H_n+\frac14n(n-1)\epsilon+O(\epsilon^2)$. [Note: This asymptotic value holds when $\epsilon$ is very very small, but it can be misleading when $\epsilon$ is larger because the error term is really $O(n^4\epsilon^2)$. Indeed, a closer analysis shows that when $(\ln n)^2/n^2\leqslant\epsilon\leqslant1$ we have $M_n(\epsilon)=\Theta(n\sqrt{\epsilon}\,)$. To prove this, we can break the sums into two ranges, using one estimate for $j\leqslant\sqrt{1/\epsilon}$ and another estimate for the larger values of $j$ (when $\rho^{\binom{j+1}{2}}$ is getting small).]

**F4** (a) Let $\alpha$ be such that $|S_{n-1} - \ln S_n| \leqslant \alpha$, and let $\beta$ be any number greater than $\alpha$. We will use the fact that $x \leqslant y - \beta$ implies $e^{x+\alpha} \leqslant e^y - \beta$ if $y$ is sufficiently large, namely if $e^y \geqslant \beta/(1 - e^{\alpha-\beta})$. Take $N$ large enough that $S_N \geqslant \ln\bigl(\beta/(1-e^{\alpha-\beta})\bigr)$ and define $t$ by the relation $S_N = e{\uparrow\uparrow}(N+t) - \beta$. Then we can prove by induction that $S_n \leqslant e{\uparrow\uparrow}(n+t) - \beta$ for all $n \geqslant N$. Similarly, if we choose $M$ and $u$ so that $S_M = e{\uparrow\uparrow}(M+u) + \beta \geqslant \ln\bigl(\beta/(e^{\beta-\alpha} - 1)\bigr)$, then $S_n \geqslant e{\uparrow\uparrow}(n+u) + \beta$ for all $n \geqslant M$. Hence

$$e{\uparrow\uparrow}(n+u) \;\leqslant\; S_n \;\leqslant\; e{\uparrow\uparrow}(n+t)$$

for all large $n$; QED.

(b) Let $\ln A_n / \ln A_{n-1} = A_{n-2}(1 - \epsilon_n)$. The crude inequalities

$$\left(\frac{m-k}{k}\right)^k \;\leqslant\; \binom{m}{k} \;\leqslant\; m^k$$

tell us that $\epsilon_n \geqslant 0$ and that

$$\begin{aligned}\epsilon_n \;&\leqslant\; \frac{\ln A_{n-2}}{\ln A_{n-1}} - \frac{\ln(1 - A_{n-2}/A_{n-1})}{\ln A_{n-1}} \\ &=\; \frac{1}{A_{n-3}(1-\epsilon_{n-1})} + O\left(\frac{A_{n-2}}{A_{n-1}\ln A_{n-1}}\right).\end{aligned}$$

We can use bootstrapping to prove that $\epsilon_n$ is very small. First we prove inductively that $A_n \geqslant 2^n$ and that $A_n > 2A_{n-1}$ for all $n > 2$. Hence $\epsilon_n \leqslant 1/\bigl(A_{n-3}(1-\epsilon_{n-1})\bigr) + O(1/n)$, hence $\epsilon_n = O(1/n)$, hence $\epsilon_n = O(1/A_{n-3})$.

(The estimate $A_n \geqslant 2^n$ is "a bit conservative," but we need to start the proof somewhere. In fact, the sequence continues after $A_1$ with

$$\begin{aligned}A_2 \;&=\; 6\,,\\ A_3 \;&=\; 15\,,\\ A_4 \;&=\; 5005\,,\\ A_5 \;&=\; 23197529289205687077586038842122627336104000\,,\end{aligned}$$

and $A_6 \approx 8.2 \times 10^{200699}$ is too large to write down here.)

Now we have

$$\begin{aligned}\ln A_n \;=\; (\ln A_1)\prod_{k=2}^{n}\frac{\ln A_k}{\ln A_{k-1}} \;&=\; (\ln 4)\prod_{k=2}^{n} A_{k-2}(1-\epsilon_k) \\ &=\; \frac{C\prod_{k=0}^{n-2} A_k}{\prod_{k>n}(1-\epsilon_k)}\end{aligned}$$

where $C = (\ln 4) \prod_{k=2}^{\infty}(1-\epsilon_k)$. (This infinite product converges because we know that $\epsilon_k = O(2^{-k})$; in fact it converges very rapidly, and we have $C \approx 0.12824$.) Notice that $\ln \prod_{k>n}(1-\epsilon_k) = \sum_{k>n} \ln(1-\epsilon_k) = O\bigl(\sum_{k>n} \epsilon_k\bigr) = O(1/A_{n-2})$, hence

$$\begin{aligned}
\ln A_n &= CA_0 \ldots A_{n-2}\bigl(1 + O(1/A_{n-2})\bigr)\,;\\
\ln\ln A_n &= \ln A_{n-2} + \ln\ln A_{n-1} + O(1/A_{n-2})\\
&= \ln A_{n-2} + O(n \log A_{n-3})\\
&= (\ln A_{n-2})\bigl(1 + O(n/A_{n-4})\bigr)\,;\\
\ln\ln\ln A_n &= \ln\ln A_{n-2} + O(n/A_{n-4})\,.
\end{aligned}$$

*Merry Christmas and Happy New Year to all!*

Now apply part (a) with $S_n = \ln\ln A_{\lfloor n/2 \rfloor}$.