☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ ERRATA TO VOLUME 4 FASCICLE 2 ☆☆☆☆

This document is a transcript of the notes that I have been making in my personal copy of *The Art of Computer Programming*, Volume 4, Fascicle 2, since it was first printed in 2005.

Three levels of updates — "errors," "amendments," and "plans" — appear, indicated by three different typographic conventions:

▸**Page 666** line 1 ________________________________ 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \rightsquigarrow n+1$

Page 666 line 2 ________________________________ 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . .' leading to the date on which I recorded the plan in my files.

My personal file of updates also includes a fourth category of items, not shown in this list. They are miscellaneous minor corrections or improvements that most readers don't want to know about, because they are so trivial. If you really want to see all of the gory details, you can download the full list from Internet webpage

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

by selecting the "long form" of the errata.

The ultimate, glorious, future editions of Volumes 1–5 are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 4B, Stanford University, Stanford CA 94305-9045, or by email to `taocp@cs.stanford.edu`*. (Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although I'm working full time on*

Volume 4 these days, I will try to reply to all such messages within a year of receipt. Current news about The Art of Computer Programming *is posted on*

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

and updated regularly.

—Don Knuth, February 2005

For a work of such scope as this,
the first word of the author should be an apology
for what is doubtless the too ambitious effort
of a single writer.

— EDWARD W. BYRN (1900)

GENERATING ALL TUPLES AND PERMUTATIONS

Last updated 3 December 2010

Most of these corrections have already been made in recent printings.

Important note: The changes below include nearly every difference between the paperback fascicle and the first printing of Volume 4A, published in January 2011. All subsequent changes can be found in the errata list for that volume.

Page iii line 12 ——— 15 Sep 2005

There will also be a Fascicle 1 for Volume 4. $\rightsquigarrow$ There will also be a Fascicle 1 for Volume 4, and even a Fascicle 0.

Page 0 replacement for lines 3 and 4 ——— 11 Oct 2008

The opening sections of this chapter can be found in Volume 4, Fascicle 0, and Volume 4, Fascicle 1, published in 2008 and 2009.

▸**Page 2** in the running headline (also on pages 4, 6, 8, ...!) ——— 29 Jan 2008

SEARCHING $\rightsquigarrow$ SEARCHING (F2)

▸**Page 3** (and throughout the fascicle!) ——— 04 Apr 2008

Fig. 10 $\rightsquigarrow$ **Fig. 30** [and all figure numbers need to increased by 20]

Page 4 line 16 ——— 18 Oct 2006

7.1–00 $\rightsquigarrow$ 7.1.3–117

Page 6 line 4 ——— 18 Oct 2006

7.1–(00) $\rightsquigarrow$ 7.1.3–(44)

Page 8 in (17) ——— 19 Apr 2009

$k = (b_{n-1} \ldots b_1 b_0)_2 \rightsquigarrow k = (\ldots b_2 b_1 b_0)_2$

Page 9 line −7 ——— 18 Oct 2006

Section 7.1 $\rightsquigarrow$ Section 7.1.3

Page 11 in step W1 ——— 18 Oct 2006

line 2: $\bigl((z-m) \oplus z \oplus m\bigr) \wedge m' \neq 0 \rightsquigarrow \bigl((z-m) \oplus z \oplus m\bigr) \mathbin{\&} m' \neq 0$
line 4: 7.1–(00); it $\rightsquigarrow$ 7.1.3–(90). It

Page 11 lines 1–2 of step W2 ——— 19 May 2005

$m_0 \leftarrow z \wedge (2^5-1)$, ... $m_4 \leftarrow z \wedge (2^{25}-2^{20}) \rightsquigarrow m_0 \leftarrow z \mathbin{\&} (2^5-1)$, $m_1 \leftarrow z \mathbin{\&} (2^{10}-2^5)$, $m_2 \leftarrow z \mathbin{\&} (2^{15}-2^{10})$, $m_3 \leftarrow z \mathbin{\&} (2^{20}-2^{15})$, and $m_4 \leftarrow z \mathbin{\&} (2^{25}-2^{20})$

Page 13 line −7 ——— 26 Mar 2009

45 $\rightsquigarrow$ 47

▸**Page 15** line −12 ——— 05 Sep 2008

$k < 2^n \rightsquigarrow k < 2^n - 1$

▸**Page 20** bottom line ______ 17 Apr 2005

jth forest $\leadsto$ jth tree

▸**Page 23** in Table 1 ______ 02 Mar 2009

$22 : 1, 7 \leadsto 22 : 1$

▸**Page 25** line 12 ______ 19 Apr 2009

step D6 $\leadsto$ **step D5**

Page 26 line 9 ______ 08 Oct 2010

68 (1958) $\leadsto$ (2) **68** (1958)

Page 30 line 1 ______ 05 Jul 2007

[*21*] $\leadsto$ [*23*]

▸**Page 31** replacement for line 4 of exercise 29 ______ 23 Mar 2009

$$\frac{1}{2^n}\sum_{k=0}^{2^n-1}\Bigl|g^{[-1]}\bigl(g(k)\oplus p\bigr)-k\Bigr|,$$

Page 32 line 2 of exercise 40 ______ 02 Mar 2009

$m_j = x \wedge (2^{5j}-1) \leadsto m_j = z \mathbin{\&} (2^{5j}-1)$

Page 33 replacement for exercises 44–48 ______ 26 Mar 2009

44. [*M20*] Show that $d(n) \le \binom{M(n)}{2}$, if the n-cube has $M(n)$ perfect matchings.

45. [*M40*] (T. Feder and C. Subi, 2009.) This exercise constructs a large number of Gray cycles in the $(4r+2)$-cube $G = G_4 \square G_3 \square G_2 \square G_1 \square G_0 \square G_{-1}$, where G_i is an r-cube for $i > 0$ and $G_0 = G_{-1} = P_2$. The vertices v are $(4r+2)$-bit strings $v_4 \ldots v_0 v_{-1}$, where v_i has r bits for $i > 0$ and 1 bit for $i \le 0$. The "signature" of v is the 4-bit string $\sigma(v) = s_4 s_3 s_2 (s_1 \oplus v_0)$, where s_i is the parity of v_i. We treat bit strings as binary numbers.

For $1 \le l \le 4$, let $\mathcal{M}_l(v)$ be a matching in G with $v \mathbin{—} v' = v'_4 \ldots v'_0 v'_{-1}$ and $v'_i = v_i$ for $i \ne l$. (Note that $\mathcal{M}_l(v') = v$.) Also define $\mathcal{M}_0(v) = v \oplus 2$. Consider the cycles formed by the edges $v \mathbin{—} \mathcal{M}_{l(v)}(v)$, where $l(v)$ depends on v's signature:

$\sigma(v) =$	0000	0001	0011	0010	0110	0111	0101	0100	1100	1101	1111	1110	1010	1011	1001	1000
$l(v) =$	0	2	0	3	1	2	0	4	1	2	1	3	1	2	0	4

a) Suppose $r = 2$ and $\mathcal{M}_l(v) = v \oplus 2^{2l+s_{l-1}}$ for $l > 1$ and $\mathcal{M}_1(v) = v \oplus 2^{2+(v_0 \oplus v_{-1})}$. What cycle contains vertex $0\ldots0$ in this case?

b) A vertex whose signature is a power of 2 is called a "ground vertex." Four vertices with the same $v_4 \ldots v_1$ are called "siblings." Define $u \equiv v$ if u and v are in the same cycle, or if u and v are sibling ground vertices, or if a chain of such equivalences leads from u to v. Explain how to construct cycles in G for each equivalence class.

c) Furthermore, if u and v are sibling ground vertices, there is such a cycle that retains the edges $\{u\oplus 2 \mathbin{—} u, v\oplus 2 \mathbin{—} v\}$ of the original cycles.

d) Finally, show how to convert the cycles of (b) and (c) into a single cycle.

e) When $\mathcal{M}_1, \ldots, \mathcal{M}_4$ vary, how many different Hamiltonian cycles do we get?

46. [*M23*] Extend exercise 45 to the $(kr+2)$-cube, for k even.

47. [*HM24*] What asymptotic estimates do exercises 44 and 46 give for $d(n)^{1/2^n}$?

48. [*HM48*] Determine the asymptotic behavior of $d(n)^{1/2^n}$ as $n \to \infty$.

Page 33 new rating for exercise 55 — 28 Aug 2007

[*47*] $\rightsquigarrow$ [*35*]

Page 36 in exercise 85 — 02 Feb 2007

$\boxtimes \rightsquigarrow \wr$ [This notational change also affects answers 85–87.]

Page 37 replacement for lines 1–4 of exercise 96 — 19 Jan 2010

▸**96.** [*M28*] Suppose a family of coroutines has been set up to generate a de Bruijn cycle of length m^n using Algorithms R and D, based recursively on simple coroutines like Algorithm S for the base case $n = 2$, and using Algorithm D when $n > 2$ is even.

a) How many coroutines (R_n, D_n, S_n) of each type will there be?

Page 38 in exercise 107 — 25 Apr 2009

running time of Algorithm F. $\rightsquigarrow$ running time of Algorithm F, for fixed m as $n \to \infty$.

▸**Page 52** line 6 after the illustration — 19 Apr 2009

from node $c_{n-1} \ldots c_{k+1} j$ to its right sibling $c_{n-1} \ldots c_{k+1}(j+1)$ $\rightsquigarrow$
from node $c_{n-1} \ldots c_{k+1}(j-1)$ to its right sibling $c_{n-1} \ldots c_{k+1} j$

Page 61 line 7 of (45) — 19 May 2005

$t \leftarrow c \wedge (11100)_2 \rightsquigarrow t \leftarrow c \mathbin{\&} (11100)_2$

Page 61 line 6 of (46) — 19 May 2005

$a \leftarrow a \vee t \rightsquigarrow a \leftarrow a \mid t$

Page 63 line 4 of Algorithm V — 22 Mar 2010

$a_0 = 0 \rightsquigarrow a_0 = a_0' = 0$

Page 67 replacement for exercise 31 in Volume 4A — 02 Jan 2010

31. [*M22*] (Nob Yoshigahara.) (a) What is the unique solution to $\mathtt{A}/\mathtt{BC}+\mathtt{D}/\mathtt{EF}+\mathtt{G}/\mathtt{HI} = 1$, when $\{\mathtt{A}, \ldots, \mathtt{I}\} = \{1, \ldots, 9\}$? (b) Similarly, make $\mathtt{AB} \bmod 2 = 0$, $\mathtt{ABC} \bmod 3 = 0$, etc.

Page 68 line 2 of exercise 45 — 16 Oct 2009

314157700 $\rightsquigarrow$ 161800000

▸**Page 71** replacement for line 8 — 11 Oct 2009

$$x \leftarrow a_{j-c_j+s+[o_j=+1]}, \qquad y \leftarrow a_{j-c_j+s-[o_j=-1]};$$

▸**Page 71** last line of exercise 69 — 05 Sep 2009

$A_n = y$. $\rightsquigarrow$ $A_n = y$; step Z4 should terminate when $A_1 \ldots A_n = n \ldots 1$.

Page 72 line 1 of exercise 75 — 20 Dec 2008

M25 $\rightsquigarrow$ *M26*

Page 72 line 1 of exercise 75 — 11 Oct 2008

directed torus $C_m \times C_n \rightsquigarrow$ directed torus $C_m^{\to} \times C_n^{\to}$

Page 75 replacement for exercise 112 ——— 03 Feb 2008

▸**112.** [*M30*] (A. Williams, 2007.) Continuing exercise 111, construct *explicit* cycles:

a) Show that a universal cycle of permutations is equivalent to a Hamiltonian cycle on the Cayley graph with two generators $\rho = (1\ 2\ \ldots\ n{-}1)$ and $\sigma = (1\ 2\ \ldots\ n)$.

b) Prove that any Hamiltonian path in that graph is actually a Hamiltonian cycle.

c) Find such a path of the form $\sigma^2\rho^{n-3}\alpha_1 \ldots \sigma^2\rho^{n-3}\alpha_{(n-1)!}$, $\alpha_j \in \{\rho, \sigma\}$, for $n \geq 3$.

113. [*HM43*] Exactly how many universal cycles exist, for permutations of ≤ 9 objects?

▸**Page 76** running headline should be blank ——— 11 Oct 2008

[delete '7.2.1.2']

Page 76 for greater elegance in answer 5 line 4 ——— 09 Jun 2005

$j > k \rightsquigarrow j < k$

▸**Page 78** line 2 of answer 16 ——— 19 Mar 2009

$g(b_1 \ldots b_n) = a_1 \ldots a_n \rightsquigarrow g(a_1 \ldots a_n) = b_1 \ldots b_n$

▸**Page 78** line 2 of answer 18 ——— 26 Aug 2007

IEEE Trans. $\rightsquigarrow$ *IRE Trans.*

Page 80 line 1 ——— 18 Oct 2006

7.1–00 $\rightsquigarrow$ 7.1.3–3

▸**Page 80** line 6 ——— 19 Mar 2009

$p = \lceil 2^t/3 \rceil \rightsquigarrow 2^{\lfloor \lg p \rfloor + 1} - p = \lceil 2^t/3 \rceil$

Page 80 last line of answer 25 ——— 17 Sep 2005

596.] $\rightsquigarrow$ 596. The analogous bound for the modular m-ary Gray code is $\lceil m^t/(m^2{-}1) \rceil$, and this formula holds also for the reflected m-ary Gray code when m is even; see van Zanten and Suparta, *IEEE Trans.* **IT-49** (2003), 485–487; *Proc. South East Asian Math. Soc. Conf.* (Yogyakarta: Gadjah Mada University, 2003), 98–105.]

▸**Page 81** line 3 of answer 30 ——— 23 Mar 2007

$b_{j+l+1} \rightsquigarrow b_{j+2^l}$

Page 81 in answer 30 ——— 19 May 2005

line 11: $r \leftarrow (r-1) \wedge m \rightsquigarrow r \leftarrow (r-1)\ \&\ m$
line 11: 7.1–00 $\rightsquigarrow$ 7.1.3–79
line 13: $m \leftarrow 2m + [t \wedge (t-1) \neq 0] \rightsquigarrow m \leftarrow 2m + [t\ \&\ (t-1) \neq 0]$

▸**Page 81** or 82, line 1 of answer 34 ——— 22 Jan 2006

$r_{j+1}^{b_j} \rightsquigarrow r_{j+1}^{b_j}(x)$

Page 82, line 6 of answer 35 ——— 03 Apr 2007

Kronecker product $\rightsquigarrow$ direct product

▸**Page 82,** line 3 of answer 37 ——— 23 Mar 2009

$\dfrac{2j+1}{2^e} \rightsquigarrow \dfrac{2j+1}{2^{e+1}}$

▸**Page 82** line 1 of answer 39 ——— 22 Jan 2006

Let's $\rightsquigarrow$ (a) Let's

▸**Page 82** line 4 of answer 39 ——— 13 Apr 2009

$(j+j')\cdot(p(j)+p(j'))$ ⤳ $(j\oplus j')\cdot(p(j)\oplus p(j'))$

Page 83 line 5 ——— 02 May 2005

145–205. ⤳ 145–205; **42** (2005), 213, 229–243. See also J. H. Conway and D. A. Smith, *On Quaternions and Octonions* (2003).

Page 84 replacement for answers 44–48 ——— 26 Mar 2009

44. Every n-bit Gray cycle defines a pair of matchings (see exercise 55).

45. (a) (`000 002 012 010 090 094 0b4` ... `112 102 100`), in hexadecimal, 32 elements in all. Notice that the signatures of elements in each cycle run through the Gray code Γ_4.

(b) A ground vertex v is preceded in its cycle by its sibling $v\oplus 2$. If v is a ground vertex in a different cycle from its sibling $u=v\oplus 1$, we can join the cycles by deleting $\{u\oplus 2 — u,\ v\oplus 2 — v\}$ and inserting $\{u — v,\ u\oplus 2 — v\oplus 2\}$. Repeat for all ground v.

(c) Consider the multigraph G' whose vertices are the cycles and whose edges go from the cycle of v to the cycle of $v+1$ for all even ground vertices v. Every vertex of G' has even degree, so the edges are a union of cycles in G'. Thus any edge of G' can be deleted without changing the connected components.

(d) It's not difficult to construct a path $P=v^{(0)} — v^{(1)} — v^{(2)} — \cdots$ through vertices of G with $v_0=v_{-1}=0$ that passes through all such v with $\sigma(v)\le 1$, and such that $\sigma(v^{(i)})\in\{0,1,2,4,8\}$ for all i. Take the cycle from (b) that contains $v^{(0)}$ and call it the "working cycle" W; then do the following for $i=1,\,2,\,\ldots,$ until W includes all vertices: If $v=v^{(i)}\notin W$, suppose $u=v^{(i-1)}$ has $u_l\ne v_l$. *Case 1:* $u\oplus c — u$ is an edge of W, for $c=1$ or $c=2$. Take a cycle for the equivalence class of v that has the edge $v\oplus c — v$. Delete those edges and insert $\{u — v,\ u\oplus c — v\oplus c\}$. *Case 2:* Otherwise Case 1 must have applied to $w=v^{(i-2)}$ and u on the previous step. If $c=1$ then W contains the edge $u\oplus 2 — u\oplus 3$. We can find a cycle with $v\oplus 2 — v\oplus 3$, and replace those edges by $\{u\oplus 2 — v\oplus 3,\ u\oplus 3 — v\oplus 3\}$. A similar edge-swap works when $c=2$.

(e) The final cycle W allows us to reconstruct $\mathcal{M}_{l(v)}(v)$. When $l(v)\ne 0$ the function $\mathcal{M}_{l(v)}$ is equivalent to $t=2^{3r-1}$ independent matchings of the r-cube, because there are t ways to choose the v_i for $i\ne l$ having the correct signature. So the number of different cycles is at least $M(r)^{12t}$ (see exercise 44).

46. There are k-bit signatures $\sigma(v)$. When $\sigma(v)=g(j)$ in Gray binary code, $l(v)=\bigl(\rho(j+1)+[j\ne 2^k-1]\bigr)[j+2$ is not a power of 2]. At least $M(r)^{(2^k-k)t}$ cycles arise, where $t=2^{(k-1)(r-1)+2}$. [*Information Processing Letters* **109** (2009), 267–272.]

47. The bounds $(\frac{r}{e})^{2^{r-1}}<2^{r-1}!/(2^{r-1}/r)^{2^{r-1}}\le M(r)\le r!^{2^{r-1}/r}=(\frac{r}{e}+O(\log r))^{2^{r-1}}$ are proved in Section 7.5.1. Hence $d(n)^{1/2^n}\le n/e+O(\log n)$ by exercise 44.

The lower bound from exercise 46, if we let G_j be an r_j-cube, is

$$\bigl(M(r_1)^{2^{n-r_1-k+1}}\bigr)^{2^{k-1}-k}\cdot\bigl(M(r_2)^{2^{n-r_2-k+1}}\bigr)^{2^{k-2}}\cdot\bigl(M(r_3)^{2^{n-r_3-k+1}}\bigr)^{2^{k-3}}\cdot\ldots\cdot\bigl(M(r_{k-1})^{2^{n-r_{k-1}-k+1}}\bigr)^2\cdot\bigl(M(r_k)^{2^{n-r_k-k+1}}\bigr)^2;$$

and it's better to choose $r_j\approx(n-2)/2^{j-[j=k]}$ for $1\le j\le k$ instead of using cubes of roughly the same size. Let $\alpha_j=r_j/e$ be a lower bound on $M(r_j)^{2^{1-r_j}}$. The lower bound on $d(n)^{1/2^n}$ simplifies to

$$\alpha_1^{1/2-k/2^k}\alpha_2^{1/4}\alpha_3^{1/8}\ldots\alpha_{k-1}^{1/2^{k-1}}\alpha_k^{1/2^{k-1}}=2^{-2+(k-4)/2^k}\Bigl(\frac{n-2}{e}\Bigr)^{1-k/2^k}\Bigl(1+O\Bigl(\frac{k}{n}\Bigr)\Bigr),$$

and this is $n/(4e)+O(\log n)^2$ when $k=\lg n+O(1)$.

Page 85 replacement for answer 51 ——— 18 Sep 2005

51. Let $c_j = 2\lfloor(2^{n-1}+j)/n\rfloor$ and $c'_j = 2\lfloor(2^{n+1}+j)/(n+2)\rfloor$. If $n \neq 3$, it is not difficult to verify that $4c_j \geq 8\lfloor 2^{n-1}/n\rfloor > 2\lceil 2^{n+1}/(n+2)\rceil \geq c'_k$ for $0 \leq j < n$ and $0 \leq k < n+2$. Therefore we can apply Theorem D to any n-bit Gray cycle with transition counts c_j, underlining b_j copies of j and putting an underlined digit 0 last, where $b_j = 2c_j - \frac{1}{2}c'_{(j+2+d) \bmod (n+2)} - [j=0]$ and d is chosen so that $c'_d = c'_{d+1}$. This construction works because $l = b_0 + \cdots + b_{n-1} = 2(c_0 + \cdots + c_{n-1}) - \frac{1}{2}(c'_0 + \cdots + c'_{n+1} - c'_d - c'_{d+1}) - 1 = c'_d - 1$ is odd. [Corollary B was discovered by T. Bakos in the 1950s, and proved in detail by A. Ádám in *Truth Functions* (Budapest: 1968), 28–37. Ádám's book also presents a proof by G. Pollák that, in fact, $c'_0 = c'_1$ for all n; hence we may take $d = 0$. See also J. P. Robinson and M. Cohn, *IEEE Trans.* **C-30** (1981), 17–23.]

Page 85 new answer ——— 23 Oct 2009

55. In fact, given any perfect matching Q of K_{2^n}, one can find in $O(2^n)$ steps a perfect matching R of the n-cube such that $Q \cup R$ is a Hamiltonian cycle of K_{2^n}. [See J. Fink, *J. Comb. Theory* **B97** (2007), 1074–1076; *Elect. Notes Disc. Math.* **29** (2007), 345–351.]

▸**Page 86** last line of answer 59 ——— 13 Apr 2009

$\{v_{k-1}, v_k, v_{k+1}, v_{k+2}\} \rightsquigarrow \{v_k, v_{k+1}, v_{k+2}, v_{k+3}\}$

▸**Page 87** in answer 64 ——— 28 Feb 2010

line −4: $-\lfloor 4\lg(m+n) - 1 + \epsilon\rfloor \rightsquigarrow -\lfloor 4\lg N - 1 + \epsilon\rfloor$
line −2: $\epsilon = 4\lg(2m/(m+n)) < 1 \rightsquigarrow \epsilon = 4\lg(2m/N) < 1 + [N = m+n]$

▸**Page 88** first line of answer 67 ——— 13 Apr 2009

$u_{2^n-1} \rightsquigarrow u_{2^{n-1}-1}$

▸**Page 89** line 4 of answer 74 ——— 16 Apr 2009

$3 \cdot 2^n/\sqrt{2\pi n} \rightsquigarrow 3 \cdot 2^{n+1}/\sqrt{2\pi n}$

Page 90 replacement for answer 75 ——— 02 Jun 2009

75. The trend-free canonical delta sequences all turn out to yield Gray *cycles*:

0 1 2 3 0 1 2 4 2 1 0 3 2 1 0 1 2 1 0 3 2 1 0 4 0 1 2 3 0 1 2 (1)
0 1 2 3 0 1 2 4 2 1 0 3 2 1 0 1 3 0 1 2 3 0 1 4 1 0 3 2 1 0 3 (1)
0 1 2 3 0 1 2 4 2 1 0 3 2 1 0 2 0 3 2 1 0 3 2 4 2 3 0 1 2 3 0 (2)
0 1 2 3 0 1 2 4 2 1 0 3 2 1 0 2 1 2 3 0 1 2 3 4 3 2 1 0 3 2 1 (2)
0 1 2 3 0 1 2 4 2 3 0 1 2 3 0 2 0 1 2 3 0 1 2 4 2 3 0 1 2 3 0 (2)
0 1 2 3 4 1 0 1 2 1 0 3 0 1 4 3 2 1 0 3 0 1 4 1 0 1 2 3 4 1 0 (3)

(The second and fourth of these are cyclically equivalent.)

▸**Page 90** line 3 of answer 78 ——— 25 Apr 2009

if $b_j < m_j \rightsquigarrow$ if $b_j < m_j - 1$

▸**Page 93** line 6 of answer 87 ——— 02 Jun 2009

$\Sigma_m \wr (T_0, \ldots, T_{m-1}) \rightsquigarrow \Sigma_m \wr (T_0, \ldots, T_{m-1})^R$

▸**Page 95** last line of answer 95 ——— 05 Nov 2005

extra bits $\rightsquigarrow$ extra digits

Page 95 replacement for lines 1–3 of answer 96 ______ 17 Jun 2005

96. (a) For example, when $n = 5$ the top-level coroutine of type R invokes a coroutine of type D for $n = 4$, which invokes two of type S for $n = 2$; hence $R_5 = D_5 = 1$ and $S_5 = 2$. The recurrences $R_2 = 0$, $R_{2n+1} = 1 + R_{2n}$, $R_{2n} = 2R_n$, $D_2 = 0$, $D_{2n+1} = D_{2n} = 1 + 2D_n$, $S_2 = 1$, $S_{2n+1} = S_{2n} = 2S_n$ have the solution $R_n = n - 2S_n$, $D_n = S_n - 1$, $S_n = 2^{\lfloor \lg n \rfloor - 1}$. Thus $R_n + D_n + S_n = n - 1$.

▸**Page 95** replacement for last paragraph of answer 96 ______ 05 Nov 2005

From these observations we can conclude that no exceptions arise at any level when the coroutine for an m^n-cycle produces the final digit of a run x^n, or the first digit following such a run. Hence the worst case occurs when the top-level coroutine activates a subcoroutine twice, making $2\lfloor \lg n \rfloor + 2\nu(n) - 3$ activations altogether.

▸**Page 95** lines 5–8 of answer 97 ______ 02 Jun 2009

$\Sigma(k+1) \rightsquigarrow \Sigma f_n(k+1)$
$\Sigma(k-1) \rightsquigarrow \Sigma f_n(k-1)$
$\Sigma(k\pm 1) \rightsquigarrow \Sigma f_n(k\pm 1)$
$\Sigma(k) \rightsquigarrow \Sigma f_n(k)$

▸**Page 96** line 8 of answer 98 ______ 02 Jun 2009

$\Sigma f_n(k) = \Sigma f_n(k \bmod 2^n) \rightsquigarrow \Sigma f_n(k) \equiv \Sigma f_n(k \bmod 2^n)$

▸**Page 96** line 11 of answer 98 ______ 22 Jan 2006

$1{+}k{+}\Sigma c_n(k{-}2) \rightsquigarrow 1{+}k{+}\Sigma\Sigma c_n(k{-}2)$

▸**Page 96** replacement for last two lines of answer 98 ______ 02 Jun 2009

(Incidentally, if $k = 2^{s+1} - 1 - (2^{s-e_1} + 2^{s-e_2} + \cdots + 2^{s-e_t})$ where $0 < e_1 < \cdots < e_t$, we have $S(k) = s + 1 + e_t + 2e_{t-1} + 4e_{t-2} + \cdots + 2^{t-1}e_1$.)

Page 99 line 10 of answer 110 ______ 06 Apr 2005

Zoltán Tóth $\rightsquigarrow$ Antal Iványi and Zoltán Tóth

Page 100 end of answer 111 ______ 08 Sep 2007

.] $\rightsquigarrow$; and exercise 7.2.1.6–122.]

▸**Page 101** line 13 of answer 1 ______ 03 Feb 2005

Reverse $a_{j+1} \ldots a_{n-1} \rightsquigarrow$ Reverse $a_{j+2} \ldots a_{n-1}$

Page 101 line 4 of answer 3 ______ 31 Dec 2005

Nārāyaṇa $\rightsquigarrow$ Śārṅgadeva and Nārāyaṇa

▸**Page 102** line 2 of answer 6(b) ______ 03 Feb 2005

$\sum_{1\le k<l\le t}(n_k - n'_k)[n_l < n'_l] \rightsquigarrow \sum_{1\le k<l\le t}(n_k - n'_k)[n_l > n'_l]$

▸**Page 102** replacement for line 2 of answer 7 ______ 03 Feb 2005

$$\frac{n_1 + \cdots + n_t}{n} + \frac{(n_1n_2 + n_1n_3 + \cdots + n_{t-1}n_t) + n_1(n_1{-}1) + \cdots + n_t(n_t{-}1)}{n(n-1)} + \cdots .$$

Page 102 insert a new sentence at the beginning of answer 9 ______ 06 Apr 2006

Assume that $r > 0$ and that we begin with $a_0 < a_1 \le a_2 \le \cdots \le a_n$.

▸**Page 103** line 1 of answer 11 ——— 05 Jan 2010

$n! + x_n, n!,$ ⇝ $n! + x_n, n! - 1,$

▸**Page 103** in answer 15 ——— 05 Jan 2010

line 2: $b_k = k - 1$ ⇝ $b_{n-k} = k - 1$
line 3: $j < k \le m$ ⇝ $j < k \le n$

▸**Page 103** line 6 of answer 17 ——— 02 Aug 2005

$d_j \leftarrow -1$ ⇝ $o_j \leftarrow -1$

▸**Page 104** line 3 of answer 20 ——— 29 Sep 2006

$2n - 1$ operations ⇝ $2n + 1$ operations

Page 104 in answer 22 ——— 05 Jan 2010

$(\mathtt{XY})_b + (\mathtt{XX})_b = (\mathtt{XYX})_b$ ⇝ $(\mathtt{X})_b + (\mathtt{X})_b = (\mathtt{XY})_b$

Page 105 new sentence for end of answer 27 ——— 29 Nov 2009

J. A. Brown, J. Szabó, and T. J. Trowbridge suggest `GREAT` + `GREAT` = `LARGE` and `GREAT` + `GREAT` = `SMALL`.

Page 105 new paragraph for answer 31 ——— 02 Oct 2010

(b) `ABCDEFGHI` = 381654729, via Algorithm X in 10 Kμ.
[*Note:* Also introduce line 1 of the answer with '(a)'. The answer to part (b) appears only the hardcopy Volume 4A, not in the paperback fascicle.]

▸**Page 106** in answer 33 ——— 05 Jan 2010

line 6: (125) ⇝ (389)
line 7: 6443 ⇝ 12221

Page 106 line 1 of answer 34 ——— 20 Oct 2010

= 100000. ⇝ = 1014 + 255 + 98731 = 100000.

▸**Page 106** in answer 36 ——— 20 Oct 2010

line 3: (`17e4`) ⇝ (`17e8`)
line 4: (`69`)(`be`) ⇝ (`69`)(`7d`)(`be`)

▸**Page 106** in answer 37 ——— 20 Oct 2010

line 1: $k!^{k-1}$ ways ⇝ $k!^k$ ways
line 3: $A_n = \ldots 10^{148}$. ⇝ $A_n = \prod_{k=1}^{n-1} k!^{k+1} = n!^{\binom{n+1}{2}} / \prod_{k=1}^{n} k^{\binom{k+1}{2}}$. For example, $A_{10} \approx 1.148 \times 10^{170}$.

▸**Page 107** line 4 of answer 43 ——— 06 Nov 2005

for $k \ge 4$, let ⇝ for $k \ge 4$, let $\alpha(k,1) = (k\ k{-}4)$, $\alpha(k,j) = (k\ k{-}3{-}j\ k{-}2{-}j)$ for $1 < j < k-2$, and

▸**Page 107** line 2 of answer 45 ——— 16 Oct 2009

425368917 ⇝ 273914568

▸**Page 109** line 7 of answer 55 ——— 22 Mar 2010

$\alpha_{(n+1)!-1-n}$ ⇝ $\alpha_{(n+1)!-1-n!}$

▸**Page 111** bottom line of answer 68 ———— 29 Jun 2008

17–25 ⇝ 17–21

▸**Page 111** in leftmost column of the display ———— 05 Sep 2009

15342 ⇝ 15432

Page 111 replacement for bottom line of answer 69 ———— 22 Oct 2009

Discrete Math. **309** (2009), 5501–5508].

▸**Page 112** lines 9–13 of answer 73 ———— 11 Oct 2008

and let α be ... odd number of cycles, ⇝ and let $\hat\alpha$ be the permutation of X that takes a vertex π into $\alpha\pi$; this permutation has g/a cycles of length a. Define the permutation $\hat\beta$ similarly. Then $\hat\alpha\hat\beta^-$ has g/c cycles of length c. If c is odd, any Hamiltonian cycle in the graph defines a cycle $\hat\sigma$ that contains all the vertices and satisfies the hypotheses of (a). Therefore $\hat\alpha$ and $\hat\beta$ have an odd number of cycles,

▸**Page 112** replacement for lines 4–10 of answer 74 ———— 11 Oct 2008

Let $\hat\sigma$ be a Hamiltonian cycle, as in the previous answer, and let $\hat\gamma = \hat\alpha\hat\beta^-$. If $\pi\hat\sigma = \pi\hat\alpha$ then $\pi\hat\gamma\hat\sigma$ must be $\pi\hat\gamma\hat\alpha$, because $\pi\hat\gamma\hat\beta = \pi\hat\alpha$. And if $\pi\hat\sigma = \pi\hat\beta$ then $\pi\hat\gamma\hat\sigma$ cannot be $\pi\hat\gamma\hat\alpha$, because that would imply $\pi\hat\gamma^2\hat\sigma = \pi\hat\gamma^2\hat\alpha$, ..., $\pi\hat\gamma^c\hat\sigma = \pi\hat\gamma^c\hat\alpha$. Thus the elements π, $\pi\hat\gamma$, $\pi\hat\gamma^2$, ..., all have equivalent behavior with respect to their successors in $\hat\sigma$.

When the path $\pi \longrightarrow \pi\hat\sigma \longrightarrow \cdots \longrightarrow \pi\hat\sigma^j$ has $k(j)$ steps of type $\hat\alpha$, we have $\pi\hat\sigma^j = \pi\hat\beta^j\hat\gamma^{k(j)}$. Thus $\pi\hat\sigma^{g/c} = \pi\hat\gamma^{t+k(g/c)}$, and we have $k(j+g/c) = k(j)$ for $j \ge 0$. The path returns to π for the first time in g steps if and only if $t + k(g/c)$ is relatively prime to c.

Page 113 last line of answer 75 ———— 11 Oct 2008

$C_m \times C_n$ ⇝ $C_m^{\rightarrow} \times C_n^{\rightarrow}$

▸**Page 113** last line of answer 76 ———— 17 Feb 2005

$\sum_{k=1}^{d-1}$ ⇝ $\sum_{k=0}^{d}$

Page 114 new sentence appended to answer 79 ———— 03 Feb 2008

(See the similar code in 7.1.3–(69).)

▸**Page 116** line 3 ———— 02 Dec 2006

$j \ge 1$ ⇝ $j > 1$.

▸**Page 116** line 8 ———— 26 Feb 2006

$\sum_{j+1}^{r-1}$ ⇝ $\sum_{j=1}^{r-1}$

▸**Page 116** line 9 ———— 02 Dec 2006

$F_{r+1} - 2F_j + (-1)^j F_{r+1-j}$ ⇝ $F_{r+1} - 2F_j - (-1)^j F_{r+1-j}$

Page 117 lines 3–5 of answer 100 ———— 12 Feb 2005

It appears likely ... for ⇝ A. D. King [*Discrete Math.* **306** (2006), 508–516] has shown that indecomposable permutations can be generated efficiently by making only a single transposition at each step. In fact, *adjacent* transpositions may well suffice; for example, when $n = 4$ the indecomposable permutations are

▶**Page 118** line 4 of answer 105 ———— 29 Mar 2006

ments; $\rightsquigarrow$ ments";

▶**Page 119** line 10 of answer 108 ———— 21 Mar 2007

239–241 $\rightsquigarrow$ 739–741

Page 120 new answer 109 ———— 11 Nov 2010

109. An ingenious construction by I. H. Sudborough and L. Morales [*Theoretical Comp. Sci.* **411** (2010), 3965–3970] proves that $f(n) \geq \frac{19}{128}n^2 + O(1)$.

▶**Page 120** line 5 of answer 111 ———— 29 Mar 2006

Theorem 2.3.4.2D $\rightsquigarrow$ Theorem 2.3.4.2G

▶**Page 120** line -2 of answer 111 ———— 21 Apr 2006

Eulerian trials $\rightsquigarrow$ Eulerian trails

Page 120 new answer 112 ———— 25 Jan 2010

112. (a) If the cycle is $a_1a_2\ldots$, use σ at step j if the subsequence $a_ja_{j+1}\ldots a_{j+n-1}$ is a permutation; otherwise use ρ.

(b) This statement follows immediately from exercise 72.

(c) Let $\Omega_2 = \sigma^2$, and obtain Ω_{n+1} from Ω_n by substituting $\sigma \mapsto \sigma^2\rho^{n-1}$ and $\rho \mapsto \sigma^2\rho^{n-2}\sigma$. For example, $\Omega_3 = (\sigma^2\rho)^2$ and $\Omega_4 = \bigl((\sigma^2\rho^2)^2\sigma^2\rho\sigma\bigr)^2$. Generate permutations by starting with $n\ldots21$ and applying the successive elements of Ω_n; for example, the sequence when $n = 4$ is

$$4321, 3214, 2143, 1423, 4213, 2134, 1342, 3412, 4132, 1324, 3241, 2431,$$
$$4312, 3124, 1243, 2413, 4123, 1234, 2341, 3421, 4231, 2314, 3142, 1432,$$

and the corresponding universal cycle is (432142134132431241234231). Notice that n moves cyclically in this sequence of permutations; and the permutations that begin with n correspond to the sequence obtained from Ω_{n-1}.

[See F. Ruskey and A. Williams, *ACM Trans. on Algorithms* **6** (2010), 45:1–45:12. Similar methods are said to be known to bell-ringers. Universal cycles can also be constructed explicitly for permutations of an arbitrary *multiset*, with a method analogous to 7.2.1.1–(62); see A. Williams, Ph.D. thesis (Univ. of Victoria, 2009).]

Page 120 and 121, changes to old answer 112 ———— 03 Feb 2008

[This answer is now the answer to exercise 113. In the second and third paragraphs, change 'Notice first ... Now consider' $\rightsquigarrow$ 'Consider'. Also delete the final paragraph.]

Page 122 and following ———— 07 Feb 2005

Miscellaneous changes to the existing index of Volume 4 Fascicle 2 are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.

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