☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ ERRATA TO VOLUME 4 FASCICLE 5 ☆☆☆☆

This document is a transcript of the notes that I have been making in my personal copy of *The Art of Computer Programming*, Volume 4, Fascicle 5, since it was first printed in 2019.

Four levels of updates — "errors," "amendments," "plans," and "improvements" — appear, indicated by four different typographic conventions:

▸**Page 666** line 1 ——— 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \leadsto n+1$

Page 666 line 2 ——— 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . .' leading to the date on which I recorded the plan in my files.

Page 666 line 4 ——— 10 Jan 1938

The fourth and final category — indicated by page and line number in smaller, slanted type — consists of minor corrections or improvements that most readers don't want to know about, because they are so trivial. You wouldn't even be seeing these items if you hadn't specifically chosen to print the complete errata list in all its gory details. Are you sure you wanted to do that?

The ultimate, glorious, future editions of Volumes 1–5 are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 4B, Stanford University, Stanford CA 94305-9045, or by email to `taocp@cs.stanford.edu`*. (Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although I'm working full time on*

Volume 4 these days, I will try to reply to all such messages within a year of receipt. Current news about The Art of Computer Programming *is posted on*

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

and updated regularly.

— Don Knuth, September 2019

In spite of my respect for you, I am not convinced by all of your changes.

— ERNEST ANSERMET, letter to Igor Stravinsky (28 November 1929)

For the first time in many years, I've pulled out a copy and read a few chapters to see how much my voice may have changed over time. I confess to wincing every so often at a poorly chosen word, a mangled sentence, . . . I cannot honestly say, however, that the voice in this book is not mine.

— BARACK OBAMA, *Dreams From My Father* (2004)

MATH PRELIMINARIES REDUX
INTRO TO BACKTRACKING
DANCING LINKS

Last updated 26 July 2022

Page ii line −4 ———— 10 Feb 2020

`www.pearsoned.com/permissions/` $\leadsto$
`www.pearson.com/permissions/`

Page iv line 19 ———— 09 Feb 2020

will solve the $\leadsto$ will solve many

Page viii new contents item ———— 18 Sep 2021

Breaking News 384

Page 4 line −7 ———— 26 Jul 2021

with probability q $\leadsto$ with probability $q = 1 - p$

Page 11 line −6 ———— 21 Jun 2022

$= 0.$ $\leadsto$ $= 0.$ It's asymptotically almost sure.

Page 13 in exercise 5(b) ———— 08 Feb 2021

Suppose A and B ... is odd. $\leadsto$ Suppose the random variable A takes the values $(m_0, m_2, m_4, \ldots)$ with probabilities $(q_0, q_2, q_4, \ldots)$; otherwise $A = 0$. Independently, the random variable B takes the values $(m_1, m_3, m_5, \ldots)$ with probabilities $(q_1, q_3, q_5, \ldots)$; otherwise $B = 0$.

Page 14 new rating for exercise 24 ———— 27 Oct 2020

[*HM27*] $\leadsto$ [*HM28*]

Page 18 line 3 of exercise 63 ———— 16 Aug 2020

those probabilities $\leadsto$ those nine probabilities

Page 19 new rating for exercise 72 ———— 21 Jun 2022

72. [*M21*] $\leadsto$ ▸ **72.** [*M23*]

▸**Page 23** line 6 of exercise 112 ———— 26 Feb 2020

$\min_{a\in\pi} a\pi = \infty$ $\leadsto$ $\min_{a\in A} a\pi = \infty$

Page 24 line 4 of exercise 121 ———— 31 May 2021

divergence of Y *from* X, $\leadsto$ *divergence of* X *from* Y,

▸**Page 26** bottom line ———— 06 May 2021

$p_{k-1} \le p_k \ge p_{k-1}$ $\leadsto$ $p_{k-1} \le p_k \ge p_{k+1}$

Page 27 new exercise 18 Sep 2021

135. [*HM30*] (*Baxter permutations.*) Let $p_1 \ldots p_n$ be a permutation of $\{1,\ldots,n\}$, and let $q_1 \ldots q_n$ be its inverse. These permutations are called Baxter permutations if and only if there are no indices k and l with $0 < k, l < n$ such that either ($q_{k+1} < l$ and $q_k > l$ and $p_l < k$ and $p_{l+1} > k$) or ($q_k < l$ and $q_{k+1} > l$ and $p_l > k$ and $p_{l+1} < k$).

What's a good way to count the number b_n of n-element Baxter permutations?

Page 27 new exercise 26 Sep 2021

136. [*HM20*] Let $f(x) = x \ln x$ be the fundamental convex function that underlies formulas for entropy. Prove or disprove: If $0 \le x \le y \le 1$ then $|f(y)-f(x)| \le |f(y-x)|$.

Page 30 line 2 14 Nov 2020

level 5 $\leadsto$ level 5 below the root.

Page 35 lines −19 and −17 15 May 2020

line −19: four-letter words $\leadsto$ four-letter codewords
line −17: from words of the set $\leadsto$ from strings of the set

Page 36 lines 2 and 3 15 May 2020

line 2: Such a word $\leadsto$ Such a string
line 3: *aperiodic* words $\leadsto$ *aperiodic* strings

Page 37 line 15 14 Nov 2020

no commafree code $\leadsto$ no commafree code of length $(m^4 - m^2)/4$

Page 39 new copy for two lines after (23) 16 Aug 2020

(This data-structuring technique has been called a *sparse-set representation*; see P. Briggs and L. Torczon, *ACM Letters Prog. Lang. and Syst.* **2** (1993), 59–69.)

Page 39 line 10 16 Aug 2020

codes is not $\leadsto$ codes of maximum length is not

▸**Page 40** line 5 after the table 30 May 2020

$\mathtt{CLOFF} + m^4 + cl(\alpha)$. $\leadsto$ $\mathtt{S3OFF} + m^4 + s_3(\alpha)$, $\mathtt{CLOFF} + m^4 + 4cl(\alpha)$.

▸**Page 43** line 2 30 May 2020

we must backtrack $\leadsto$ we mustn't proceed

▸**Page 44** line 2 of step C6 30 May 2020

$f \leftarrow f - 1 \leadsto f \leftarrow f + 1$

▸**Page 47** line 6 05 Jun 2020

$y_1 y_2 \ldots \leadsto y_0 y_1 \ldots$

Page 48 line 19 14 Nov 2020

shaky grounds $\leadsto$ shaky ground

Page 48 line 2 before (34) 19 Jan 2020

P. Diaconis and S. Chatterjee $\leadsto$ S. Chatterjee and P. Diaconis

Page 48 and 49 ——— 19 Jan 2020

[I'm now using the notation $\widehat{\chi}_N$ instead of Q_N, for eventual agreement with Section 7.2.2.9]

Page 51 line −16 ——— 27 Oct 2021

a student named $\rightsquigarrow$ a telegraph engineer named

Page 53 replacement for exercise 15 ——— 19 Dec 2021

15. [*HM42*] (M. Simkin, 2021.) Show that $Q(n) \approx \sigma^n n!$ as $n \to \infty$, where $\sigma \approx 0.389068$.

Page 54 new rating for exercise 19 ——— 14 Nov 2020

[*M19*] $\rightsquigarrow$ [*M10*]

▸**Page 55** line 9 of exercise 37 ——— 22 Dec 2020

$x_k > x_{k+1} \le x_{k+2} \rightsquigarrow x_k \ge x_{k+1} < x_{k+2}$

Page 56 new rating for exercise 46 ——— 17 Aug 2020

[*M30*] $\rightsquigarrow$ [*M35*]

Page 58 new rating for exercise 69 ——— 05 Dec 2019

[*40*] $\rightsquigarrow$ [*41*]

▸**Page 59** in 9(D) of Table 666 ——— 24 Oct 2020

$[61\,..\,67] \rightsquigarrow [59\,..\,67]$

Page 60 in lines 4 and 6 of exercise 77 ——— 01 Jun 2021

$P_3^{\to} \mathbin{\square} P_3^{\to} \rightsquigarrow P_3^{\vec{}} \mathbin{\square} P_3^{\vec{}}$

Page 64 line 2 before (5) ——— 17 Aug 2020

find a subset $\rightsquigarrow$ find all subsets

Page 65, replacement for the third bullet point in (9) ——— 17 Aug 2020

- For each just-deleted option O that involves i, one at a time, do this:
  for each item $j \neq i$ in O, cover item j; then
  recursively append O to each solution of the residual problem.

Page 66 just before Table 1 ——— 01 Jan 2020

are unused. $\rightsquigarrow$ are unused and can contain anything.

Page 66 first line of cover(i) ——— 01 Jan 2020

(Here p, l, and r are local variables.) $\rightsquigarrow$ (Undeclared variables like p, l, r are local.)

Page 71 line 6 ——— 29 Nov 2019

amendment of $\rightsquigarrow$ amendment to

Page 78 line 6 before (37) ——— 21 Nov 2020

consecutive letters. $\rightsquigarrow$ consecutive letters from O to Z.

▸**Page 84** line 24 ——— 22 Jun 2021

'`4 000 011 111 121 4-`' $\rightsquigarrow$ '`4 001 011 111 121 4-`'

Page 84 line −2 ———— 21 Dec 2019

Exercises 325–345 ⇝ Exercises 324–350

Page 87 line 4 after (49) ———— 27 Mar 2020

never examined. ⇝ never examined, except in printouts (see the answer to exercise 12).

Page 87 line 7 of Table 2 ———— 19 Sep 2021

`LEN(`x`)` ⇝ `LEN(`x`),TOP(`x`)`

Page 87 new line for nodes $x = 0, 1, 2, 3, 4, 5, 6$ in Table 2 ———— 27 Mar 2020

`COLOR(`x`):`	—	—	—	—	—	—	0

Page 88 in first line of (55) ———— 27 Mar 2020

$q \leftarrow$ `DLINK(`i`)`. ⇝ `COLOR(`i`)` $\leftarrow c$, $q \leftarrow$ `DLINK(`i`)`.

Page 88 third line of step C5 ———— 15 Jul 2020

This covers ⇝ This commits

Page 88 third line of step C6 ———— 15 Jul 2020

This uncovers ⇝ This uncommits

Page 96 line 1 ———— 15 Jul 2020

partially covers ⇝ commits

Page 96 line 4 of step M7 ———— 15 Jul 2020

uncovers ⇝ uncovers or uncommits

▸**Page 100** line 9 ———— 11 Jul 2022

(see exercise 188) ⇝ (see exercise 189)

▸**Page 102** line 6 ———— 10 Jul 2021

evenually ⇝ eventually

▸**Page 107** line 19 ———— 07 Mar 2020

'$a\ g$' and '$a\ f$'. Further simplifications now arise. ⇝ '$a\ g$' and '$a\ f$'.

Page 110 line 1 after (108) ———— 23 Dec 2021

both Σ_1 and Σ_2. ⇝ both Σ_1 and Σ_2!

▸**Page 113** line 7 ———— 26 Feb 2022

belongs of ⇝ belongs to

Page 122 last two lines of exercise 2 ———— 24 Oct 2020

say $(a_1, \ldots, a_k)$, $\ldots (a_1, \ldots, a_k)$. ⇝
say $a_1, \ldots, a_k$, then undelete them in exactly the same order $a_1, \ldots, a_k$.

▸**Page 126** in exercise 49 ———— 09 Aug 2021

[the bipartite graph should have an edge between u_7 and v_1]

Page 127 last sentence of exercise 53 ________________ 21 Mar 2022

Show that … problems. $\leadsto$ Show that exactly 13 essentially different 4-clue shidoku problems have a unique solution.

▸**Page 127** line 1 of exercise 57 ________________ 10 Sep 2021

has 27 $\leadsto$ has at most

Page 130 last line of exercise 76 ________________ 19 Sep 2021

an identity element $\leadsto$ the identity element 0

Page 130 an additional sentence in exercise 77 ________________ 02 Mar 2020

(In such a case we say that G is *embedded* in H.)

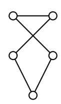
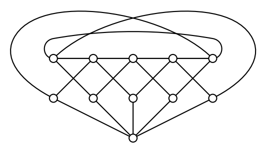
Page 133 in exercise 103(b) ________________ 30 Mar 2022

How many solutions $\leadsto$ How many inequivalent solutions

Page 135 line 1 of exercise 116 ________________ 25 May 2021

Given a graph on vertices V $\leadsto$ Given a graph G on vertices V

Page 135 better illustrations in exercise 116 ________________ 19 Feb 2020

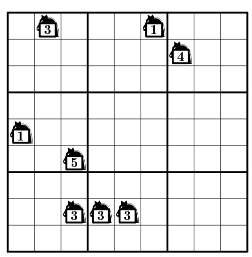
M_2 = ; M_3 =  ; M_4 =  ; . . . .

▸**Page 136** in exercise 123 ________________ 23 Dec 2021

line 5: into m parts $\leadsto$ into parts
line 6: $+\cdots{n \brace m}m^{\underline{m}}$ $\leadsto$ $+\cdots+{n \brace m}m^{\underline{m}}$

▸**Page 144** correction to puzzle (ii) in exercise 173 ________________ 04 Dec 2019

… Diagram (ii) shows 8 … the other 19 can be deduced. …



▸**Page 144** rewording of exercise 177 ________________ 17 Nov 2021

177. [*M21*] Algorithm 7.2.1.5M generates the $p(n_1,\ldots,n_m)$ partitions of the multiset $\{n_1\cdot x_1,\ldots,n_m\cdot x_m\}$ into submultisets. Consider the special cases where $n_1=\cdots=n_s=1$ and $n_{s+1}=\cdots=n_{s+t}=2$ and $s+t=m$.

a) Generate those partitions with Algorithm M, using the previous two exercises.

b) Also generate the $q(n_1,\ldots,n_m)$ multipartitions into *distinct* multisets.

Page 148 in exercise 208 ________________ 19 Dec 2021

‘`1a 2a 3a 4a 2b` $V_{\tt 1b}$’ and ‘`1c 2c 3c 4c 3b` $V_{\tt 1b}$’ $\leadsto$ ‘`1c 2c 3c 4c 3b` $V_{\tt 1b}$’ and ‘`1a 2a 3a 4a 2b` $V_{\tt 1b}$’

Page 150 in exercise 225 ——— 08 Dec 2021

the 6×15 case $\rightsquigarrow$ the "one-sided" 6×15 case

▸**Page 150** in exercise 235 ——— 18 Oct 2020

costs \$$d(i,j)^4$ $\rightsquigarrow$ costs \$$4d(i,j)^4$

▸**Page 152** line 2 of exercise 252 ——— 25 Jul 2022

`ROOT(0)` $\rightsquigarrow$ `RLINK(0)`

▸**Page 155** line 1 of exercise 281 ——— 02 Sep 2020

$(2n{+}1 \times (2n{+}1)$ $\rightsquigarrow$ $(2n{+}1) \times (2n{+}1)$

Page 155 line 3 of exercise 284 ——— 25 May 2021

(a) $\{I, P, U, V\}$ and (b) $\{F, P, T, U\}$ $\rightsquigarrow$ (a) $\{\mathrm{O}, \mathrm{P}, \mathrm{U}, \mathrm{V}\}$ and (b) $\{\mathrm{P}, \mathrm{R}, \mathrm{T}, \mathrm{U}\}$

▸**Page 161** replacement for lines 4–6 of exercise 139 ——— 08 Feb 2022

there are three diaboloes: {□, ▱, △}. Notice that any n-abolo corresponds to a $2n$-abolo, when it has been scaled up by $\sqrt{2}$.

Page 161 changes to exercise 322 ——— 07 Jun 2022

For example, ... white. $\rightsquigarrow$ For example, there are two disticks and five tristicks; and of course there's only one monostick. They're shown here in white, surrounded by the sixteen tetrasticks in black.

[And of course there's a new illustration, with colors swapped.]

▸**Page 164** in exercise 337 ——— 17 Mar 2022

line 4: right-handed $\rightsquigarrow$ left-handed
line 5: left-handed $\rightsquigarrow$ right-handed

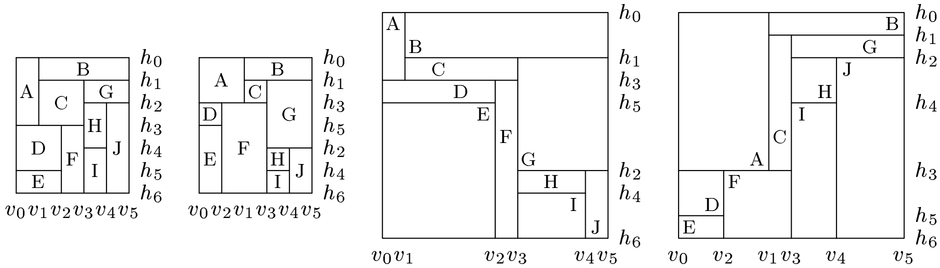
▸**Page 165** the illustrations in exercise 341(a) and 345 ——— 25 May 2021

[the darker cubies should be much darker; compare with the colored versions online in `cs.stanford.edu/~knuth/fasc5c.ps.gz`, page 103]

Page 169 and 170, replacement for exercise 372 ——— 10 Sep 2021

▸**372.** [*M35*] (*Floorplans.*) If a rectangle decomposition satisfies the tatami condition — "no four rectangles meet" — it's often called a *floorplan*, and its subrectangles are called *rooms*. The line segments that delimit rooms are called *bounds*. Four possibilities arise when room r is adjacent to bound s: Either $s \downarrow r$, $r \to s$, $r \downarrow s$, or $s \to r$, meaning respectively that the top, right, bottom, or left boundary of r is part of s.

For example, the floorplans shown on the next page have 10 rooms $\{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{J}\}$, $7+6$ bounds $\{h_0, \ldots, h_6, v_0, \ldots, v_5\}$, and the following adjacencies: $h_0 \downarrow \mathrm{A} \downarrow h_3 \downarrow \mathrm{D} \downarrow h_5 \downarrow \mathrm{E} \downarrow h_6$, $h_0 \downarrow \mathrm{B} \downarrow h_1 \downarrow \mathrm{C} \downarrow h_3 \downarrow \mathrm{F} \downarrow h_6$, $h_1 \downarrow \mathrm{G} \downarrow h_2 \downarrow \mathrm{H} \downarrow h_4 \downarrow \mathrm{I} \downarrow h_6$, $h_2 \downarrow \mathrm{J} \downarrow h_6$; $v_0 \to \mathrm{A} \to v_1 \to \mathrm{B} \to v_5$, $v_1 \to \mathrm{C} \to v_3 \to \mathrm{H} \to v_4$, $v_0 \to \mathrm{D} \to v_2 \to \mathrm{F} \to v_3 \to \mathrm{G} \to v_5$, $v_0 \to \mathrm{E} \to v_2$, $v_3 \to \mathrm{I} \to v_4 \to \mathrm{J} \to v_5$.



Two floorplans with the same adjacencies are considered to be equivalent. Thus, all four of the floorplans above are essentially the same, even though they look rather different: In particular, room C needn't overlap room D; we require only $\mathrm{C} \downarrow h_3 \downarrow \mathrm{D}$.

a) Let $r \Downarrow r'$ mean that $r = r_0 \downarrow s_0 \downarrow r_1 \downarrow \cdots \downarrow s_{k-1} \downarrow r_k = r'$ for some $k > 0$; define $r \Rightarrow r'$ similarly. Prove that $[r \Downarrow r'] + [r \Rightarrow r'] + [r' \Downarrow r] + [r' \Rightarrow r] = 1$, when $r \neq r'$. *Hint:* Every floorplan has unique diagonal and antidiagonal equivalents, as shown.

b) A *twin tree* is a data structure whose nodes v have four pointer fields, `L0`(v), `R0`(v), `L1`(v), `R1`(v). It defines two binary trees T_0 and T_1 on the nodes, where T_θ is rooted at `ROOT`θ and has child links (`L`θ, `R`θ). These two trees satisfy inorder$(T_0) =$ inorder$(T_1) = v_1 \ldots v_n$; `R0`$(v_k) = \Lambda \iff$ `R1`$(v_k) \neq \Lambda$, for $1 \leq k < n$.
For each room r, if r's top left corner is a $\top$ junction, set `L0`$(r) \leftarrow \Lambda$ and `L1`$(r) \leftarrow r'$, where r' is the room opposite r in that corner; otherwise reverse the roles of `L0` and `L1`. Similarly, set `R0`$(r) \leftarrow \Lambda$ and `R1`$(r) \leftarrow r'$ if the bottom right corner of r is a $\dashv$ junction, or vice versa otherwise. (Use $r' = \Lambda$ at extreme corners.) Also set `ROOT0` and `ROOT1` to the bottom-left and top-right rooms. Show that a twin tree is created, convenient for representing this floorplan.

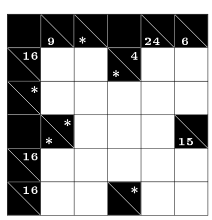
Page 170 and 171, in exercise 375 ——— 10 Feb 2020

smallest possible perimeter ⇝ smallest possible semiperimeter (height plus width)
smallest perimeter ⇝ smallest semiperimeter

Page 172, append new sentence to exercise 387 ——— 27 Oct 2020

(Note that mirror reflection is *not* a legal symmetry for a polycube; L-twist $\neq$ R-twist!)

▸**Page 180** corrected diagram in exercise 433 ——— 25 May 2021



Page 183 last line of answer 2 ——— 26 Aug 2021

156.] ⇝ A. Komisarski, *AMM* **128** (2021), 423–434.]

▸**Page 185** replacement for lines 3 and 4 of answer 21 ——— 11 Jul 2022

probability 1/5. Then $\mathrm{E}\,XY = \mathrm{E}\,X = 3/5$ and $\mathrm{E}\,Y = 1$; $\Pr(X=0)\Pr(Y=1) = 2/25 \neq 0$.

Page 185 in answer 24 ——— 18 Oct 2020 Nobel Agrawal Boyd

line 2 becomes

$$B'_{m,n}(x) = \sum_{k=1}^{m} \binom{n}{k} k x^{k-1}(1-x)^{n-k} - \sum_{k=0}^{m} \binom{n}{k} (n-k) x^k (1-x)^{n-1-k}$$

line 7: It suffices $\rightsquigarrow$ If $a > 0$ it suffices
line 12: increases $\rightsquigarrow$ is nondecreasing

Page 186 last line of answer 24 ——— 24 Jul 2022

332.] $\rightsquigarrow$ 332. See also S. Janson, *Statistics and Probability Letters* **171** (2021) 109020, 10 pages.]

Page 191 new answer 63 ——— 16 Aug 2020

63. If ω is the event '$Z_0 = a$ and $Z_1 = b$', we have $Z_0(\omega) = a$ and $\mathrm{E}(Z_1 \mid Z_0)(\omega) = (p_{a1} + 2p_{a2})/(p_{a0} + p_{a1} + p_{a2})$. Hence $p_{01} = p_{02} = p_{20} = p_{21} = 0$, and $p_{10} = p_{12}$. Those conditions are necessary and sufficient for $\mathrm{E}(Z_1 \mid Z_0) = Z_0$.

Page 193 first line of answer 72 ——— 21 Jun 2022

all paths $\rightsquigarrow$ all paths to (r, b)

▸**Page 194** in answer 83 ——— 16 Aug 2020

line 2: $\min(m, N)$ $\rightsquigarrow$ $N'_n(x_0, \ldots, x_{n-1}) = [n < m] \cdot N_n(x_0, \ldots, x_{n-1})$
line 3: S_n $\rightsquigarrow$ S_N

Page 196 in answer 101 ——— 09 Jun 2022

By 1.2.10–(25), therefore, and setting $t = \theta/\mu$, $\rightsquigarrow$ Set $t = \theta/\mu$, and note that $p_k \ln(1 - t/p_k) \ge p \ln(1 - t/p) \ge \frac{1}{\mu} \ln(1 - t\mu) = \frac{t}{\theta} \ln(1 - \theta)$. By 1.2.10–(25), therefore,

Page 200 lines 7 and 8 of answer 114 ——— 05 Jan 2022

we could find subsets ... avoids them all $\rightsquigarrow$ we could eliminate them one (or more) at a time, by removing an element from any S_j for which $|S_j| > d_j + 1$

▸**Page 205** line 4 of answer 133 ——— 06 Jul 2022

rows, by induction $\rightsquigarrow$ distinct columns, by induction

Page 205 new answer 135 ——— 18 Sep 2021

135, 136, ... See page 384.

▸**Page 206** lines 4 and 5 of answer 10 ——— 22 Dec 2020

$a_l \leftarrow a_{l-1} + t$ ... and $\rightsquigarrow$
$a_{l+1} \leftarrow a_l + t$, $b_{l+1} \leftarrow (b_l + t) \gg 1$, $c_{l+1} \leftarrow ((c_l + t) \ll 1)\ \&\ \mu$; and

▸**Page 207** line 2 of answer 12 ——— 11 Oct 2020

we have also $\sum_{k=1}^{n} k \equiv \binom{n+1}{2}$ $\rightsquigarrow$ the same sum is also $\equiv \binom{n+1}{2}$

Page 207 replacement for answer 15 ——— 19 Dec 2021

15. More precisely, there's a constant $\sigma = e^{1-\alpha}$ such that, for any fixed ϵ with $0 < \epsilon < \sigma$, $Q(n)/n!$ is q.s. between $((1-\epsilon)\sigma)^n$ and $((1+\epsilon)\sigma)^n$. In fact, a subtle analysis [arXiv:2107.13460 [math.CO] (2021), 51 pages] shows that the average of all solutions approaches a fascinating probability distribution. P. Nobel, A. Agrawal, and S. Boyd have computed α accurately [arXiv:2112.03336 [math.CO] (2021), 14 pages].

▶**Page 208** line 3 of answer 24 — 22 Apr 2020

word x_3 ⇝ word x_l

▶**Page 211** line 7 of answer 37 — 22 Dec 2020

$b_{e+1} = a_e b_e b_e$ ⇝ $b_{e+1} = b_e a_e b_e$

▶**Page 211** line 1 of step O3 — 22 Dec 2020

less $(j-1) = 1$ ⇝ less$(j-1) = 1$

▶**Page 211** in step O4 — 22 Dec 2020

If $j < t$, set $b'_{t'} \leftarrow b_j$; ⇝ If $i < t$, set $b'_{t'} \leftarrow b_i$;
$b'_0 \leftarrow b_{j-t}$ ⇝ $b'_0 \leftarrow b_{i-t}$

▶**Page 211** line 3 of step O5 — 18 Dec 2019

while $b_{k-t} < n$ ⇝ while $b_{k-t} < 2n$

▶**Page 212** line 2 of answer 38 — 22 Dec 2020

$z_2 = j$ ⇝ $z_{k-1} = j$

▶**Page 212** line 1 of answer 44 — 30 May 2020

$r \leftarrow m+1$ ⇝ $r \leftarrow 5$

▶**Page 213** line -5 of answer 45 — 30 May 2020

go to C6 ⇝ go to C5

Page 213 replacement for last paragraph of answer 46 — 08 Feb 2021

So the answer is "No." But we can come close: Aaron Windsor used a SAT solver in 2021 to discover a binary commafree code for $n = 12$ that contains a representative of every cycle class except [000011001011].

▶**Page 213** line 8 of answer 47 — 30 May 2020

1002, 2201, 2001, ⇝ 1002, 1102, 2001,

Page 214 replacement for answer 48 — 08 Feb 2021

48. Algorithm C isn't fast enough to solve this problem. But Aaron Windsor used a SAT solver in 2021 to find such a code of size $139 = (5^4 - 5^2)/4 - 11$, and to prove that no such code of size 140 exists. (He also found, rather quickly, that an optimum ternary commafree code for $n = 6$ contains $(3^6 - 3^3 - 3^2 + 3^1)/6 - 3 = 113$ codewords.)

Page 217 line -3 of answer 61 — 18 Sep 2021

#R44. ⇝ #R44, 1–16.

▶**Page 221** line 3 of answer 73 — 12 Dec 2019

`roots` ⇝ `roofs`

▶**Page 222** line 1 of step R6 — 12 May 2020

set $i \leftarrow i_l$ ⇝ set $i \leftarrow i_l$, $v \leftarrow v_i$

▶**Page 222** in step R2' — 05 Feb 2020

$p \geq L$ ⇝ $w \geq L$

▸**Page 223** line 19 — 09 Apr 2020

$(1 + 3wz^2 + 2w^2z^3 + z^2z^4 + w^3z^4)^8 \rightsquigarrow (1 + 3wz^2 + 2w^2z^3 + w^2z^4 + w^3z^4)^8$

▸**Page 224** replacement for lines 1–6 of answer 2 — 24 Oct 2020

2. (a) (Solution by X. Lou.) We can prove that LLINK(a_k) $= a_k - 1$ and RLINK(a_k) $=$ $(a_k+1) \bmod (n+1)$ when a_k is undeleted; hence that undeletion sets RLINK($a_k - 1$) and LLINK($(a_k + 1) \bmod (n + 1)$) to the correct value a_k. (If $a_k - 1$ wasn't deleted *before* a_k, LLINK(a_k) never changed. Otherwise LLINK(a_k) became $a_k - 1$ when $a_k - 1$ was undeleted, by induction on k. A similar argument works for RLINK.) Notice that each LLINK and RLINK is reset exactly once, except that LLINK(1) and RLINK(n) remain 0.

Page 226 line 2 of answer 12 — 30 Jul 2021

NAME(TOP(q)) $\rightsquigarrow$ 'NAME(TOP(q))'

Page 226 append new paragraph to answer 12 — 27 Mar 2020

[Algorithm C extends Algorithm X to colors. If COLOR(q) $\neq 0$, also print '$:c$' where $c =$ COLOR(q) if COLOR(q) > 0, otherwise $c =$ COLOR(TOP(q)).]

▸**Page 229** last line of answer 27 — 06 Aug 2021

5 4 6 10 12 5 $\rightsquigarrow$ 5 4 6 8 10 12 5

▸**Page 229** last line of answer 29 — 14 Dec 2019

$\{1, 2\} \rightsquigarrow \{0, 1\}$. [Also change $^{01}_{10}$ to $^{10}_{01}$ in lines 4–5, 9–10, 13–14 of the matrix.]

Page 231 last line of answer 37 — 03 Mar 2021

Combinatorics, to appear. $\rightsquigarrow$ *Combinatorics* **27** (2020), #P1.52, 1–27.

Page 231 replacement for step G3 in answer 38 — 07 Feb 2021

G3. [Found?] If $k > n - r$ go to G4. Otherwise if $a_k = b_{k+n} = c_{k-n} = 0$, go to G5. Otherwise set $k \leftarrow k + 1$ and repeat this step.

Page 234 replacement for answer 52 — 14 Dec 2019

52. (Solution by F. Stappers.) Puzzles claiming to be "the world's hardest sudoku" keep appearing in online forums. Rated by search tree size with Algorithm X, the toughest among nearly 27,000 such extreme puzzles is shown here in a canonical form. (It's number 6539 in a list available from `sites.google.com/site/sudoeleven/` (2011).) Its randomized search tree sizes are 24400 ± 1900 — astonishingly high for sudoku; and its mean running time is about 12 Mμ.)

1	2		3			4		
5			4			1		
				2			6	
7								
			7				3	1
				5	4	7		
4			5			3		
	8							
		9		4				

▸**Page 236** lines 6 and 7 — 05 Feb 2021

There are only two solutions ... symmetric. $\rightsquigarrow$ Include '$*_j{:}k$' in option $(0, j, k)$. There are only four solutions, found in 3 Gμ, centrally symmetric and reducing under transposition to only two.

▸**Page 237** bottom line — 29 Jun 2022

in the SUDOKU answer, swap D $\leftrightarrow$ U in the top row

▸**Page 239** last line of answer 70 — 25 May 2021

63]; $\rightsquigarrow$ 63;

▸**Page 239** line −3 of answer 72 ———— 04 Jul 2020

1292687 packings $\leadsto$ 1292697 packings

Page 240 replacement for last line of answer 74 ———— 18 Dec 2021

[See `www.solitairelaboratory.com/puzzlelaboratory/DominoGG.html`.]

▸**Page 241** line 2 of answer 77 ———— 02 Mar 2020

each pair of vertices $\leadsto$ each pair of edges

▸**Page 241** in the table for answer 80, to match page 87 ———— 27 Mar 2020

LEN(x) $\leadsto$ LEN(x),TOP(x)
[the entries for COLOR(x) in nodes $x = 6, 11, 16, 19, 22, 25$ should be zero, not '—'.]

▸**Page 241** line 2 of answer 83 ———— 01 Jan 2020

$j > N \leadsto j > N_1$

Page 242 line 6 of answer 87 ———— 25 May 2021

items w' and appending $c_1 \ldots c_n{}'$ $\leadsto$ items w@ and appending $c_1 \ldots c_n$@

▸**Page 243** in answer 90 ———— 18 Apr 2021

line 12: `LOVED|GIVES` $\leadsto$ `LOVED|GIVEN`
line 13: `BONES` $\leadsto$ `TONES` and `WHILE` $\leadsto$ `WHOLE`

Page 244 in answer 91 ———— 06 Jun 2021

line −8: the shortest $\leadsto$ an optimum
line −5: `FALLS` $\leadsto$ `WALLS`

Page 244 line 2 of answer 92 ———— 25 May 2021

w, w' $\leadsto$ w, w@

Page 245 near the top ———— 25 May 2021

line 1: $c_1 \ldots c'_n$' $\leadsto$ $c_1 \ldots c_n$@'
line 3: items w' $\leadsto$ items w@

▸**Page 247** and 248, replacement for first paragraph of answer 103(e) ———— 30 Mar 2022

(e) The solution for $n = 2$ has every possible symmetry; and both solutions x for $n = 4$ are equivalent to x^R, $-x^Q$, and $-x^{QR}$. But for $n > 4$ one can show that x is equivalent to at most one of the $4\varphi(n)$ rows cx, cx^R, cx^Q, cx^{QR} besides itself. We obviously can't have $x \equiv cx$ when $c \neq 1$. An elementary but nontrivial proof shows also that $x \equiv cx^R$ implies $c \bmod n = 1$; $x \equiv cx^Q$ implies $c \bmod n = n-1$; $x \equiv cx^{QR}$ implies $c \bmod n = n/2 + 1$ and $n \bmod 8 = 4$. (See Richard Stong in *AMM*, to appear.) Gilbert stated incorrectly [page 196] that no solutions of the latter kind exist; he had overlooked 12-tone rows such as 0 3 9 1 2 4 11 8 7 5 10 6, 0 1 4 9 3 11 10 8 5 7 2 6, 0 1 8 11 10 3 9 5 7 4 2 6, for which $x \equiv 7x^{QR}$. Similarly, the 20-tone row 0 1 3 11 2 19 13 9 12 7 14 18 4 17 16 8 6 15 5 10 satisfies $x \equiv 11x^{QR}$.)

▸**Page 248** last line of answer 103 ———— 30 Mar 2022

$5x^{QR} \leadsto 7x^{QR}$

▸**Page 248** line 1 of answer 104 ———— 16 Dec 2019

We may $\leadsto$ (a) We may

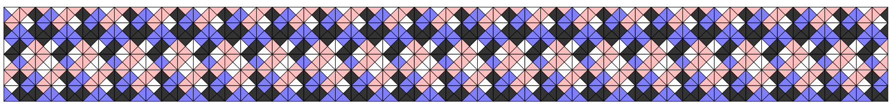
Page 249 lines 6, 7, 10, 15, 16, 17 of answer 109 ______ 25 May 2021

[enclose each option in single quotes; for example, line 6 should begin with '#k and line 7 should end with `0`']

Page 250 new copy following the displayed solutions in answer 109 ______ 16 Sep 2021

Suppose there are w words of total length s. Aaron Windsor suggests adding options 'E ij:. ij':0' for $1 \le i \le m$ and $1 \le j \le n$, where E is a new primary item representing an empty cell. All solutions to the MCC problem with the number of E's in the interval $[mn - s + w - 1 \,..\, mn]$ are then either connected or contain a cycle.

▶**Page 253** replacement for line -3 ______ 15 Oct 2021



▶**Page 256** line -5 ______ 21 Oct 2021

`13:a` or `-13:c` $\rightsquigarrow$ `-13:a` or `-13:c`

Page 259 replacement for Q in answer 136, line 16 ______ 27 Oct 2020

$$Q = \frac{1}{2}\begin{pmatrix} -1 & -\phi & 1/\phi \\ -\phi & 1/\phi & -1 \\ 1/\phi & -1 & -\phi \end{pmatrix} \rightsquigarrow Q = \frac{1}{2}\begin{pmatrix} 1 & -\phi & 1/\phi \\ \phi & 1/\phi & -1 \\ 1/\phi & 1 & \phi \end{pmatrix}$$

▶**Page 260** line 9 of answer 138 ______ 23 Jul 2020

$96/6 = 12$ cases $\rightsquigarrow$ $96/6 = 16$ cases

▶**Page 263** line -4 of answer 143 ______ 21 Oct 2021

$(5, 1, 1, 0)$ $\rightsquigarrow$ $(12, 12, 1, 0)$

▶**Page 263** line 5 of answer 145 ______ 24 Jul 2020

3 three odd $\rightsquigarrow$ 3 odd

▶**Page 267** line 2 of answer 152 ______ 22 Jul 2022

multiplicity 11, and without $\rightsquigarrow$ multiplicity 11, and a blank pattern of multiplicity 1, but without

Page 268 updated URL for last line of answer 154 ______ 11 Jul 2022

`erich-friedman.github.io/mathmagic/0607.html`

▶**Page 275** line 3 of answer 176 ______ 11 Jul 2022

'$\#_j B_j$' $\rightsquigarrow$ '$\#_j\ B_j$'

▶**Page 277** replacement for answer 189(a) ______ 19 Dec 2019

(a) $|e^{e^z}| = |e^{e^x \cos y + ie^x \sin y}| = \exp(e^x \cos y)$; $|e^{-e^z}| = \exp(e^{-x} \cos y)$.

▶**Page 278** line 2 of answer 198 ______ 11 Jul 2022

with $a_j \le s$ $\rightsquigarrow$ with $a_j \ge s$

Page 280 line -4 of answer 207 ______ 06 Nov 2020

96 Gμ, with a 55-meganode $\rightsquigarrow$ 92 Gμ, with a 54-meganode

▶**Page 280** line 7 of answer 208 ——— 17 Nov 2021

inserts $V_{i(j-1)}$, $H_{(i+1)j}$, $V_{i(j+1)}$, $\rightsquigarrow$ inserts V_{ij}, $H_{(i+1)j}$, $V_{i(j+1)}$, H_{ij}

▶**Page 281** in answer 215 ——— 17 Feb 2022

line 11 should be: (e) Use only (i,j,k,l) for $k = i+1 < \min(j,l)$ and i even.
line 15: $(q-1)(q-3) \rightsquigarrow (2q)(2q-2)$

▶**Page 284** replacement for answer 232 ——— 18 Oct 2020

232. No. Algorithm X$^{\$}$ finds 96 solutions of minimum cost \$84; but the true solution in Fig. 74(a) actually costs \$86 by this measure. The effects of 16 rounding errors, each potentially changing the result by nearly \$1, have invalidated everything. [Therefore the author used $\$\lfloor 2^{32} d(i,j)\rfloor$ when preparing Fig. 74. This was safe, because the distance between the first 8 solutions and the 9th was greater than 16 — in fact, *much* greater, although a difference of only 17 would have been convincing.]

▶**Page 286** in answer 239 ——— 26 Nov 2019

line 3: cost $\epsilon^i \rightsquigarrow$ cost ϵ^j
line 9: *Graphs and Algorithm* $\rightsquigarrow$ *Graphs and Algorithms*

Page 286 line 8 of answer 239 ——— 28 Nov 2019

smaller cost $\epsilon^4 \rightsquigarrow$ smaller additional cost $0+\epsilon^4$

▶**Page 288** line 3 of answer 249 ——— 26 Jan 2020

Set $l = t = 0$. $\rightsquigarrow$ Set $l \leftarrow t \leftarrow 0$.

▶**Page 288** line 4 of answer 249 ——— 08 Dec 2021

$p \leftarrow t$, $r \leftarrow 1 \rightsquigarrow p \leftarrow t$, $y \leftarrow 0$, $r \leftarrow 1$

▶**Page 290** line 2 of answer 261(b) ——— 11 Jul 2022

$S = \{s_1,\ldots,s_k\}$ and $T = \{t_1,\ldots,t_k\} \rightsquigarrow S = \{s_1,\ldots,s_m\}$ and $T = \{t_1,\ldots,t_m\}$

▶**Page 293** lines 1 and 2 ——— 28 Aug 2020

exact problem $\rightsquigarrow$ exact cover problem

▶**Page 294** line 5 of answer 270 ——— 19 Dec 2021

90 rotated $\rightsquigarrow$ 90°-rotated

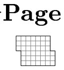
▶**Page 294** line 2 of answer 271 ——— 19 Dec 2021

1152 $\rightsquigarrow$ 152

▶**Page 298** line 2 of answer 283 ——— 02 Sep 2020

365(b) $\rightsquigarrow$ 282(b)

▶**Page 301** line 6 needs a 42-cell container ——— 27 Dec 2021



Page 302 line −5 of answer 298 ——— 25 May 2021

Supplement (1936) $\rightsquigarrow$ *Supplement* **2**, 16 (February 1936)

Page 303 last paragraph of answer 300 ——— 28 Dec 2021

line 2: "shadow" $\rightsquigarrow$ "ghost"
line 8: shadow $\rightsquigarrow$ ghost
line 15: shadows $\rightsquigarrow$ ghosts

Page 303 updated URL for last line of answer 300 — 27 Dec 2021

`erich-friedman.github.io/mathmagic/0507.html`

▶**Page 304** replacement for first paragraph of answer 303 — 06 Feb 2020

303. (a) Represent the tree as a sequence $a_0a_1\ldots a_{2n-1}$ of nested parentheses; then a_0 will match a_{2n-1}. The left boundary of the corresponding parallomino is obtained by mapping each '`(`' into N or E, according as it is immediately followed by '`(`' or '`)`'. The right boundary, similarly, maps each '`)`' into N or E according as it is immediately *preceded* by '`)`' or '`(`'. For example, if we take 7.2.1.6–(1) and enclose it in an additional pair of parentheses, the corresponding parallomino is shown below with part (d).

▶**Page 304** replacement for lines 6, 7, 8 of part (b) — 19 Jan 2020

1+1+1+1+2+2+2+2+2+2+2+2+2+1+1; there's an "outer" G, whose H is $yxyGy$, and an "inner" G, whose H is $xyyxyxxy$.) Thus we can write G as a continued fraction,

$$G(w,x,y)=wxy/(1-wx-wy-w^3xy/(1-w^2x-w^2y-w^5xy/(\cdots))).$$

▶**Page 306** four lines after the first display — 21 Aug 2021

[begin a new paragraph with the answer to part (e)]

Page 307 line 2–4 of answer 308 — 24 Jan 2022

takes $(x,y)\mapsto(x+y,x_{\max}-x)'$ ... nonnegative. $\leadsto$ takes $(x,y)\mapsto(x+y,-x)'$ and $(x,y)'\mapsto(x+y+1,-x)$; the shape's triangles should be shifted afterwards so that all coordinates are nonnegative and as small as possible.

Page 312 and many nearby pages — 13 Jun 2022

IMPORTANT NOTE: I used two different conventions while writing the exercises for Section 7.2.2.1 during a period of years. If a problem has x solutions that are essentially different, but $x\cdot y$ solutions when symmetry has not been taken into account, I would sometimes write '$x\cdot y$' solutions, and sometimes '$y\cdot x$'. From now on I shall try to be consistent, always writing '$y\cdot x$' when y is the amplification factor (the number of automorphisms divided by the number of symmetries). For example, the convex polyabolo shape $(3\times 11;2,2,1,1)$ displayed on line 12 of this page has $2\cdot 236$ solutions, not $236\cdot 2$. This change affects the answers to exercises 290, 305, 306, 309, 313, 316, 317, 319, 320, 328, 329, 330, 332, 333, 338, 339, and 391; but it leaves the answers of many other exercises unchanged, especially the earlier ones.

▶**Page 313** lines −3 and −2 of answer 322 — 25 May 2021

Dawson in the 1940s, $\leadsto$ Dawson,

Page 314 line 2 of answer 325 — 11 Mar 2022

have the tee in a fixed position and the claw restricted; $\leadsto$ that restrict the tee and the claw as suggested in the text;

Page 314 line 19 of answer 325 — 11 Nov 2021

Section 7.1.4 $\leadsto$ Section 7.1.4.2

Page 314 last lines of answer 325 — 30 Jun 2021

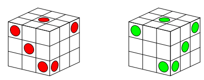
line −2: five special $\leadsto$ seven special
line −1: B5f, R7d $\leadsto$ B5f, W4e, W2f, R7d

Page 315 in last paragraph of answer 327 ——— 22 Dec 2019

Hall (clip, underpass) $\rightsquigarrow$ Hall (piano, clip, underpass)
Morgan (face, gorilla, smile) $\rightsquigarrow$ Morgan (piano, face, gorilla, smile)

Page 318 in answer 337 ——— 07 Jun 2022

[to help clarify the first paragraph, two new illustrations show the "rear view"]



Page 318 line −6 of answer 337 ——— 14 Dec 2019

[these illustrations should be darker, I've now thickened the lines]

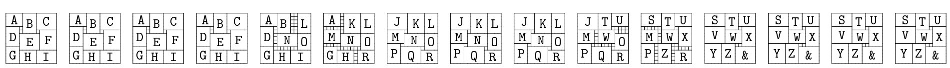
▸**Page 322** line 7 ——— 25 Dec 2019

Budapestiensis $\rightsquigarrow$ *Budapestinensis*

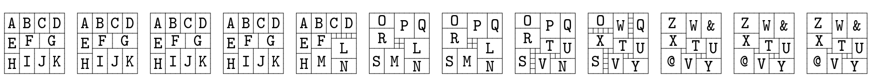
Page 322 lines −3 and −2 of answer 346 ——— 14 Dec 2019

(online 18 June 2018) $\rightsquigarrow$ **61** (2019), 271–284

Page 322 replacement for line −2 of answer 349 ——— 18 Oct 2020



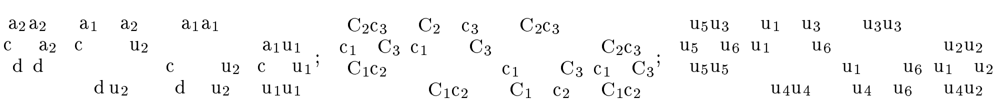
Page 322 replacement for line −3 of answer 350 ——— 18 Oct 2020



▸**Page 325** line 13 ——— 25 Sep 2020

planar places $\rightsquigarrow$ planar pieces

▸**Page 326** replacement for the display after line 3 of part (d) ——— 14 Apr 2022



▸**Page 326** line 4 of answer 357 ——— 28 Sep 2020

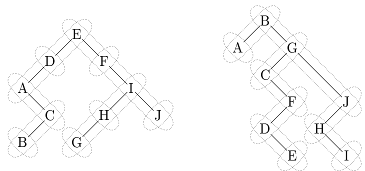
if and only the $\rightsquigarrow$ if and only if the

Page 330 and 331, replacement for answer 372 ——— 18 Sep 2021

372. (a) This fact, and the others noted below, can be proved by induction on the number of rooms: If the lower right corner of the upper left room is a $\perp$ junction, we can "flatten" and remove that room by bringing its right bound left; otherwise we can bring its bottom bound up. All floorplans can be built up by reversing this flattening process.

Let the rooms be $r_1 \ldots r_n$ in diagonal order and $r_{p_1} \ldots r_{p_n}$ in antidiagonal order (left to right). Then $r_i \Downarrow r_j \iff i < j$ and i follows j in the permutation $p = p_1 \ldots p_n$; $r_i \Rightarrow r_j \iff i < j$ and i precedes j in p. The number of horizontal bounds is the number of *descents* in p, plus 2. The number of vertical bounds is the number of *ascents*, plus 2.

(b) Here's the twin tree structure for the example. Notice that its leftward and rightward chains are the ordered sequences of rooms adjacent to the bounds.



Every twin tree structure arises in a remarkably simple way: Let $p = p_1p_2 \ldots p_n$ be any permutation of $\{1, 2, \ldots, n\}$. Obtain T_0 by inserting p_1, p_2, $\ldots$, p_n into an initially empty binary tree; obtain T_1 similarly by inserting p_n, $\ldots$, p_2, p_1. Those trees can be constructed in linear time (exercise 6.2.2–50); and it's easy to see that they are twins, both with inorder $12\ldots n$. Although different permutations can yield the same twin tree, exactly one *Baxter permutation* (exercise MPR–135) does so; and it can be computed from the twin tree in linear time(!). Thus there are nice one-to-one correspondences between floorplans, twin trees, and Baxter permutations.

[Floorplans are important in VLSI layout, where rooms correspond to modules and bounds correspond to channels. Twin trees were introduced by S. Dulucq and O. Guibert in *Discrete Math.* **157** (1996), 91–106, purely for their combinatorial interest, then applied to floorplans by B. Yao, H. Chen, C.-K. Cheng, and R. Graham in *ACM Trans. Design Aut. Electronic Syst.* **8** (2003), 55–80. See also J. M. Hart, *Int. J. Comp. Inf. Sciences* **9** (1980), 307–321; H. Murata, K. Fujiyoshi, T. Watanabe, and Y. Kajitani, *Proc. Asia South Pacific Design Aut. Conf.* **2** (1997), 625–633; E. Ackerman, G. Barequet, and R. Y. Pinter, *Discrete Applied Mathematics* **154** (2006), 1674–1684; and the author's programs FLOORPLAN-TO-TWINTREE, TWINTREE-TO-BAXTER, BAXTER-TO-FLOORPLAN, available online (2021).]

Page 332 and 333, in answer 375 ——— 10 Feb 2020

perimeter $\rightsquigarrow$ semiperimeter [throughout]

▸**Page 333** in answer 375(b) ——— 10 Feb 2020

$h_6, h_4)$ $\rightsquigarrow$ $h_6, h_5)$ [four times]

▸**Page 333** line 4 of answer 376 ——— 09 Feb 2020

$d = (w+1)/(w(w+2))$ $\rightsquigarrow$ $d = (w+2)/((w+1)(w+3))$

Page 333 line 3 of answer 378 ——— 21 Aug 2021

has a subset $\rightsquigarrow$ has a finite subset

▸**Page 334** in answer 379 ——— 25 May 2021

line 2: $h \times (w-2)$ $\rightsquigarrow$ $5 \times (w-2)$
lines 6 and 7: or ($h = 5$ and w is odd) or $\rightsquigarrow$ or

Page 335 line 4 of answer 383 ——— 30 Nov 2019

dissection, into $\rightsquigarrow$ dissection, of a $92 \times 92 \times 92$ cube into

Page 336 lines 5 and 8 ——— 25 May 2021

[these illustrations should be darker, I've now thickened the lines]

Page 340 last line of answer 398 ——— 30 Mar 2020

published by Tatsuo Yano $\rightsquigarrow$ published by Ryuoh Yano

Page 341 lines 2 and 5 of answer 401 — 25 May 2021

[these illustrations should be darker, I've now thickened the lines]

Page 344 replacement for answer 413 — 23 Dec 2019

413. (Solution simplified by R. Molinari.) Let each record for an item include two new fields `U` and `V`. The `U` and `V` fields of a secondary item that represents edge u — v will point to the primary items u and v. The `U` and `V` fields of a primary item that represents vertex v are renamed `MATE` and `INNER`. `MATE`(v) is zero until v first becomes the endpoint of an edge, after which it points to the other endpoint of the path fragment containing that edge. `INNER`(v) is nonzero when v lies within a path fragment.

Introduce two new global variables: Global variable `F` is the current number of fragments. Global variable `E` is the edge that closed a loop, or zero if there's no loop.

For example, suppose two edges currently have color 1, say v_1 — v_2 and v_3 — v_4. Then we've set `MATE`$(v_1) \leftarrow v_2$, `MATE`$(v_2) \leftarrow v_1$, `MATE`$(v_3) \leftarrow v_4$, `MATE`$(v_4) \leftarrow v_3$, and `F` $\leftarrow 2$. If now v_2 — v_5 joins the fray, we set `MATE(MATE`(u)`)` $\leftarrow u$, `MATE(MATE`(v)`)` $\leftarrow v$, and `INNER`$(v_2) \leftarrow 1$; but we leave `MATE`(v_2) unchanged. Subsequent edges to v_2 are rejected.

When the 'purify' routine (55) is called to give color 1 to a new edge i, it will refuse to do so when `E` is nonzero, because a loop has already been closed. Furthermore, when `E` $= 0$, it will know that edge i shouldn't be chosen if `U`(i) and `V`(i) are mates and `F` $\neq 1$, because that would close a loop disjoint from other fragments. On the other hand, it *will* close the loop if `F` $= 1$, also setting `E` $\leftarrow i$.

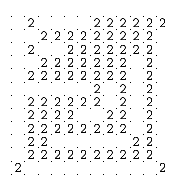
All of these operations are nicely and easily undone when we need to 'unpurify'. For example, suppose edge i loses color 1 when $u =$ `U`(i) and $v =$ `V`(i). If $v =$ `MATE`(u), we unclose the loop (and set `E` $\leftarrow 0$) if $i =$ `E`; otherwise we zero the mates and set `F` $\leftarrow$ `F` $- 1$. If `MATE`$(u) \neq$ `MATE`(v), we set `MATE`$(u) \leftarrow u$, `MATE`$(v) \leftarrow v$, `INNER`$(u) \leftarrow$ `INNER`$(v) \leftarrow 0$, `F` $\leftarrow$ `F` $+ 1$. The case `MATE`$(u) =$ `MATE`(v) is easy too.

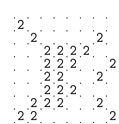
Caution: Algorithm P must be modified so that it never discards redundant items, when it is used to preprocess a problem for this extension of Algorithm C.

Page 346 additional material for answer 416 — 21 Dec 2021

But $\beta(2)$ remains unknown.

The intriguing case of $\beta(2)$ remains unknown. Beluhov proved that it is at most $\frac{11}{16}$, using a construction for $n = 4k$ that's illustrated here for $n = 12$. Palmer Mebane has constructed this 2-homogeneous puzzle on an 8×8 board that has only 24 clues [see `puzsq.jp/main/puzzle_play.php?pid=14178`].





Page 346 replacement for answer 420 — 20 Dec 2021

420. (Solution by Palmer Mebane.) In any solution, each cell is either inside or outside the loop. *Lemma: Every* 2 *has exactly two neighbors inside.* (For if the 2 is outside, the neighbors opposite its two edges are inside; otherwise the neighbors opposite its two *non*edges are inside.) Let S be the set of cells next to a 2. Color each 2 alternately red or blue; then each cell of S is a neighbor of one red 2 and one blue 2. In particular, that's true for each *inside* cell of S. Thus, by the lemma, there are equally many red and blue cells. But that contradicts $m \bmod 4 = n \bmod 4 = 1$!

[This exercise is due to N. Beluhov, who observes that solutions aplenty exist when m is odd and $n \bmod 4 = 3$.]

Page 346 (originally page 347) line −3 of answer 421 ——— 30 Mar 2020

by Tatsuo Yano $\rightsquigarrow$ by Ryuoh Yano

▸**Page 346** (originally page 347) last line of answer 421 ——— 24 Dec 2019

89 $\rightsquigarrow$ **90**

▸**Page 348** in answer 424 ——— 30 Mar 2022

lines 1–2 (or 3–4): 93659 signatures that aren't contained in any other. $\rightsquigarrow$ 93859 that aren't contained in (or equal to) the signature of any other loop.
line 6 (or 8): 1110101111010111110111011 $\rightsquigarrow$ 0000110000111011110110011

Page 348 replacement for lines 3–5 of answer 425 ——— 25 Dec 2019

not. Solutions for $k = 2$ exist for all $n \geq 3$; solutions for $k = 3$ exist for all $n \geq 4$; solutions for $k = 4$, due to B. S. Ho, exist for all $n \geq 5$; solutions for $k = 5$ and $k = 6$, due to G. J. H. Goh, exist for all $n \geq 6$. (See (xviii)–(xxiv) in Fig. A–5.) Goh has also discovered analogous constructions for $k = 7$, 8, 9, 10.

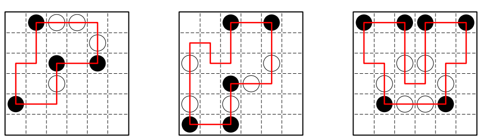
Page 348 lines 4 and 5 of answer 426 ——— 30 Mar 2020

there cannot be three consecutive '●', nor five '○'s that form an X or T. $\rightsquigarrow$ there cannot be three consecutive '●'s.

▸**Page 349** in Fig. A–5 ——— 30 Mar 2022

[flip diagrams (ii) and (iii) vertically, so that their clues are subsets of (i)]

▸**Page 349** replacement for (xx), (xxi), (xxii) in Fig. A–5 ——— 24 Dec 2019



Page 353 last line of answer 438 ——— 11 Nov 2021

Section 7.1.4 $\rightsquigarrow$ Section 7.1.4.2

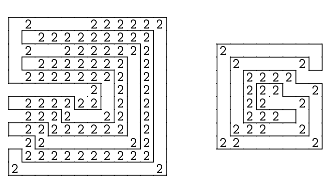
▸**Page 355** line 14 of answer 445 ——— 04 Dec 2019

an MCC problem $\rightsquigarrow$ an MCC problem

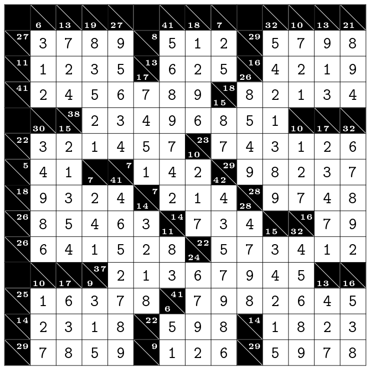
Page 356 three new lines for the end of answer 449 ——— 07 Feb 2020

The current record for largest literary hitori nugget, $12 \times 5 = 60$, was found by Gary McDonald in September 2019: "Ruth intimated that, as far as she could judge, he was a very eligible swain." [Charles Dickens, *Martin Chuzzlewit*.]

Page 359 two new answers for exercise 416 ——— 21 Dec 2021



Page 360 include a missing answer ———————————————— 25 May 2021



(see answer 431)

Page 361 and following ———— 17 Sep 2019

Miscellaneous changes to the existing index of Volume 4 Fascicle 5 are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.

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Page 384 a new section following the index ——— 18 Sep 2021

BREAKING NEWS

HERE are answers to MPR exercises that were recently added.

135. (Notice that the smallest *non*-Baxter permutations are 3142 and its inverse, 2413.)

If P is a Baxter permutation, so are $P^R = p_n \ldots p_1$ and $P^C = \bar{p}_1 \ldots \bar{p}_n$, where $\bar{x} = n+1-x$. So is the permutation $P \setminus n$ obtained by deleting n; and so are the permutations $P \setminus x$ obtained by deleting x and subtracting 1 from each element that exceeds x, if $x = p_n$ or $x = 1$ or $x = p_1$. (Consider, for example, deleting n from P^-.)

Let's look at the $n+1$ permutations obtained by *inserting* $n+1$ into an n-element Baxter permutation. For example, when $n = 8$ and $P = 21836745$ the nine extensions are 921836745, 291836745, 219836745, 218936745, 218396745, 218369745, 218367945, 218367495, 218367459. Only four of these fail Baxter's property, namely 291836745, 218396745, 218369745, and 218367495; and we soon discover the general rule: $n+1$ *can be Baxterly inserted if and only if it's placed just before a left-to-right maximum, or just after a right-to-left maximum.* (In our example, the left-to-right maxima are 2 and 8; the right-to-left maxima are 5, 7, and 8.)

Let $B_n(i,j,k)$ be the number of $(n+1)$-element Baxter permutations with exactly $i+1$ left-to-right maxima, $j+1$ left-to-right minima, k ascents, and $n-k$ descents. Such permutations correspond to floorplans with $n+1$ rooms, $i+1$ rooms touching the bottom of the frame, $j+1$ rooms touching the left of the frame, $k+2$ vertical bounds, and $n-k+2$ horizontal bounds (see exercise 7.2.2.1–372). The reasoning above yields the interesting recurrence

$$B_1(i,j,k) = [i=j=k=0],\ \ B_{n+1}(i+1,j+1,k) = \sum_{i'>i} B_n(i',j,k) + \sum_{j'>j} B_n(i,j',k-1);$$

and the solution can be expressed as a determinant of binomial coefficients:

$$B_n(i,j,k) = \det \begin{pmatrix} \binom{n-j-1}{k-1} & \binom{n}{k-1} & \binom{n-i-1}{n-k+1} \\ \binom{n-j-1}{k} & \binom{n}{k} & \binom{n-i-1}{n-k} \\ \binom{n-j-1}{k+1} & \binom{n}{k+1} & \binom{n-i-1}{n-k-1} \end{pmatrix}, \ \text{unless} \begin{cases} i=0 \text{ and } j=n \\ \quad\text{or} \\ i=n \text{ and } j=0. \end{cases}$$

Summing on i and j now gives the simpler formula

$$b_n(k) = t_{n+1}(k+1)/t_{n+1}(1), \quad \text{where } t_n(k) = \binom{n}{k-1}\binom{n}{k}\binom{n}{k+1},$$

for the number of n-element Baxter permutations with exactly k ascents.

Since the terms with $k \approx n/2$ dominate the sum $b_n = \sum_k b_n(k)$, we obtain the asymptotic value

$$b_n = \frac{8^{n+2}}{\sqrt{12}\pi n^4}\Bigl(1 - \frac{22}{3n} + O(n^{-2})\Bigr),$$

due to A. M. Odlyzko. [See G. Baxter, *Proc. American Math. Soc.* **15** (1964), 851–855; F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman, *Journal of Combinatorial Theory* **A24** (1978), 382–394; W. M. Boyce, *Houston J. Math.* **7** (1981), 175–189; S. Dulucq and O. Guibert, *Discrete Math.* **180** (1998), 143–156.] R. L. Ollerton has found the recurrence $(n+2)(n+3)b_n = (7n^2+7n-2)b_{n-1} + 8(n-1)(n-2)b_{n-2}$, with $b_1 = 1$, as well as the closed form $b_n = F\bigl({1-n,-n,-1-n \atop 2,3} \bigm| -1\bigr)$. The initial terms are $(b_0, b_1, \ldots) = (1, 1, 2, 6, 22, 92, 422, 2074, 10754, 58202, \ldots)$.

Page 384 new answer ——— 26 Sep 2021

136. It's true if $y \le x + \frac{1}{2}$, because $f(x+t) - f(x)$ increases from $f(t)$ to $-f(1-t)$ as x increases from 0 to $1-t$. But it fails when $x < \frac{1}{2}$ and $y = 1$.