☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ ERRATA TO VOLUME 4 FASCICLE 4 ☆☆☆☆

This document is a transcript of the notes that I have been making in my personal copy of *The Art of Computer Programming*, Volume 4, Fascicle 4, since it was first printed in 2006.

Four levels of updates — "errors," "amendments," "plans," and "improvements" — appear, indicated by four different typographic conventions:

▸**Page 666** line 1 ——— 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \rightsquigarrow n+1$

Page 666 line 2 ——— 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . .' leading to the date on which I recorded the plan in my files.

Page 666 line 4 ——— 10 Jan 1938

The fourth and final category — indicated by page and line number in smaller, slanted type — consists of minor corrections or improvements that most readers don't want to know about, because they are so trivial. You wouldn't even be seeing these items if you hadn't specifically chosen to print the complete errata list in all its gory details. Are you sure you wanted to do that?

The ultimate, glorious, future editions of Volumes 1–5 are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 4B, Stanford University, Stanford CA 94305-9045, or by email to `taocp@cs.stanford.edu`. *(Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although I'm working full time on*

Volume 4 these days, I will try to reply to all such messages within a year of receipt. Current news about The Art of Computer Programming is posted on

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

and updated regularly.

— Don Knuth, March 2006

He did a lot of editing. That's how he liked to work.
Sometimes he even made alterations between printings.
— P. D. JAMES, *The Lighthouse* (2005)

☆☆☆☆☆ GENERATING TREES; HISTORY ☆☆☆☆☆

Last updated 1 December 2010

Most of these corrections have already been made in recent printings.

Important note: The changes below include nearly every difference between the paperback fascicle and the first printing of Volume 4A, published in January 2011. All subsequent changes can be found in the errata list for that volume.

Page iv line 27 ———— 28 Oct 2008

I shall happily pay a finder's fee of \$2.56 $\leadsto$ I happily offer a “finder's fee” of \$2.56

Page iv lines 31 and 32 ———— 28 Oct 2008

reward you with immortal glory instead of mere money $\leadsto$ do my best to give you immortal glory

Page v replacement for lines 1–18 ———— 09 Jan 2009

A note on notation. Near the beginning of Chapter 7, some operations on graphs are defined for which many different notations are presently rampant. If G is a graph on the vertices $U = \{u_1, \ldots, u_m\}$ and if H is a graph on the vertices $V = \{v_1, \ldots, v_n\}$, larger graphs can be constructed in two “additive” and five “multiplicative” ways, as follows.

- $G \oplus H$ is the direct sum, aka juxtaposition, of G and H: It has the $m + n$ vertices $U \cup V$ and the edges of G and H.
- $G \mathbin{—} H$ is the join of G and H: On the vertices $U \cup V$, its edges are those of G and H, plus all $u_j \mathbin{—} v_k$ for $1 \le j \le m$ and $1 \le k \le n$.
- $G \mathbin{\square} H$ is the Cartesian product of G and H: It has the mn vertices $U \times V$; its edges are $(u,v) \mathbin{—} (u',v)$ when $u \mathbin{—} u'$ in G, and $(u,v) \mathbin{—} (u,v')$ when $v \mathbin{—} v'$ in H.
- $G \otimes H$ is the direct product, aka conjunction, of G and H: Again its vertices are $U \times V$, but its edges are $(u,v) \mathbin{—} (u',v')$ if and only if $u \mathbin{—} u'$ in G and $v \mathbin{—} v'$ in H.
- $G \boxtimes H$ is the strong product of G and H: As its symbol implies, it combines the edges of $G \mathbin{\square} H$ and $G \otimes H$.
- $G \mathbin{\triangle} H$ is the odd product of G and H: On the vertices $U \times V$, its edges are $(u,v) \mathbin{—} (u',v')$ if and only if $u \mathbin{—} u'$ in G or $v \mathbin{—} v'$ in H, but not both.
- $G \circ H$ is the lexicographic product, aka composition, of G and H: Again on vertices $U \times V$, it has the edge $(u,v) \mathbin{—} (u',v')$ if and only if we have either (i) $u \mathbin{—} u'$ in G or (ii) $u = u'$ and $v \mathbin{—} v'$ in H.

Page 1 replacement for line 4 ———— 11 Oct 2008

2006 and 2007 $\leadsto$ *2008 and 2009*

Page 5 line 4 — 09 Jul 2010

$2n-1$. $\rightsquigarrow$ $2n-1$. (We use a_0 as a sentinel in step P4.)

Page 6 line 4 — 29 Sep 2010

a descendant $\rightsquigarrow$ a proper descendant

▸**Page 12** line 8 — 26 Aug 2006

508–515 $\rightsquigarrow$ 508–516

▸**Page 15** (and throughout this section!) — 04 Apr 2008

Fig. 36 $\rightsquigarrow$ **Fig. 57** [and all figure numbers within section 7.2.1.6 need to increase by 21]

Page 19 line -9 — 18 Oct 2006

Section 7.1 $\rightsquigarrow$ Section 7.1.3

▸**Page 21** line -2 — 25 Jul 2006

Table 8 $\rightsquigarrow$ Table 4

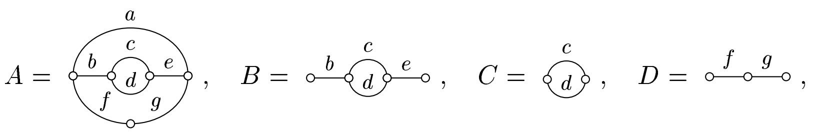
▸**Page 25** line 8 — 08 Dec 2007

(1999) $\rightsquigarrow$ (1997)

Page 25 line -9 — 21 Sep 2009

contains q $\rightsquigarrow$ contains $\{q\}$

▸**Page 28** replacement for (54) — 09 Sep 2009



(54)

Page 31 notational changes in Table 5 — 02 Feb 2007

$P_4 \times P_4 \rightsquigarrow P_4 \mathbin{\square} P_4$, $P_5 \times P_5 \rightsquigarrow P_5 \mathbin{\square} P_5$, etc.

▸**Page 31** in the column for Algorithm S in Table 5 — 09 Jul 2010

3,168.16 $\rightsquigarrow$ 3,172.19

▸**Page 31** in the column for Algorithm S′ in Table 5 — 09 Jul 2010

49.09 Kμ $\rightsquigarrow$ 49.09 Mμ

▸**Page 32** line 16 — 24 Apr 2006

137–142 $\rightsquigarrow$ 137–140

▸**Page 32** lines 19–21 — 02 Feb 2009

[The text of Theorem S should appear in *slanted* type.]

Page 32 (not an error) — 12 Jun 2008

[The index refers to a “pun resisted” on this page. I was thinking of “preposterous”. . .]

▸**Page 35** line 5 of exercise 26 ——— 18 Jul 2010

of (9) and (10) $\rightsquigarrow$ of (10) and (11)

Page 45 notational changes in exercises 105 and 106 ——— 02 Feb 2007

$G' \times G'' \rightsquigarrow G' \mathbin{\square} G''$, $G' \diamond G'' \rightsquigarrow G' \otimes G''$, $G' \mathbin{\diamond\!\!\!\times} G'' \rightsquigarrow G' \boxtimes G''$, etc. (These changes also affect the answers to exercises 105–108.)

Page 45 new parts to exercise 105 ——— 05 Apr 2007

h) $G = G' \vartriangle G''$ is the odd product of regular graphs G' and G''.
i) $G = G' \circ G''$ is the lexicographic product of regular graphs G' and G''.

[Also the rating changes from [*HM37*] to [*HM38*].]

Page 46 the rating of exercise 121 ——— 29 May 2006

[*M32*] $\rightsquigarrow$ [*M34*]

Page 50 line -5 ——— 12 Jan 2006

code-names $\rightsquigarrow$ code names

▸**Page 54** lines 21 and 23 ——— 12 Jan 2006

line 21 (in Eq. (10)): $3124 - 4321 \rightsquigarrow 3124 - 4312$
line 23: $-\,a_4b_3c_2d_1 \rightsquigarrow -\,a_4b_3c_1d_2$

Page 54 bottom line ——— 12 Jan 2006

Bhāskara $\rightsquigarrow$ BHĀSKARA

▸**Page 55** last two lines of (11) ——— 16 Oct 2010

زط and زي $\rightsquigarrow$ رط and ري

▸**Page 57** (and throughout this section!) ——— 12 Nov 2008

Fig. 44 $\rightsquigarrow$ **Fig. 64** [and all figure numbers within section 7.2.1.7 need to increase by 20]

Page 61 line 7 ——— 19 Mar 2006

radix-2 arithmetic $\rightsquigarrow$ radix-2 numbers

Page 67 line -14 or -15 ——— 29 Nov 2009

Bishop Wibold $\rightsquigarrow$ We saw above that Bishop Wibold

Page 67 line -11 ——— 18 Jan 2009

order.] $\rightsquigarrow$ order.] Thomas Harriot, in unpublished work a few years earlier, had considered up to six dice [see J. Stedall, *Historia Math.* **34** (2007), 398].

Page 70 line 13 ——— 25 Sep 2010

J. C. Burkhardt $\rightsquigarrow$ J. K. Burckhardt

▸**Page 70** lines 18–19 ——— 18 Jul 2010

Archiv für ... 194 $\rightsquigarrow$ *Archiv der reinen und angewandten Mathematik* **1** (1794), 154–195

Page 71 line −6 ——— 24 Mar 2008

Erdős $\rightsquigarrow$ Erdös [to agree with the English spelling in the paper cited]

▸**Page 74** new wording for exercise 15 ——— 19 Mar 2006

15. [*15*] If all n-combinations $x_1 \ldots x_n$ of $\{1, \ldots, m\}$ with repetition are listed in lexicographic order, with $x_1 \leq \cdots \leq x_n$, how many of them begin with the number j?

▸**Page 74** in exercise 17 ——— 12 Jan 2006

algorithm of exercise 15 $\rightsquigarrow$ algorithm of exercise 16

▸**Page 75** line 2 of exercise 28 ——— 17 Mar 2010

$bcaaa \cdot b \rightsquigarrow cbaaa \cdot b$

▸**Page 76** line 1 of answer 9 ——— 29 Sep 2010

an ancestor of node k if and only if $\rightsquigarrow$ a (proper) ancestor of node k if and only if $j < k$ and $s_j + j \geq k$.

▸**Page 78** in answer 19 ——— 20 Mar 2010

line 7: preorder $\rightsquigarrow$ postorder
line 8: if $x < n \rightsquigarrow x > 1$
line 9: $x + 1, \ldots, n \rightsquigarrow x - 1, \ldots, 1$

▸**Page 78** line −6 ——— 22 Sep 2009

make j and its siblings $\rightsquigarrow$ make j and its right siblings

▸**Page 79** line 4 ——— 20 Mar 2010

The forest F^{RTR} is called the *dual* of F; see exercise 26(f). Several $\rightsquigarrow$ Several

Page 80 line 5 ——— 01 Jan 2007

488–497] $\rightsquigarrow$ 488–497; **49** (2006), 351–357]

▸**Page 80** line 1 of answer 25 ——— 10 Jun 2010

Algorithms N and R $\rightsquigarrow$ Algorithms N and L

▸**Page 84** line 1 of answer 33(c) ——— 26 Mar 2010

$\rho\sigma\lambda\rho \rightsquigarrow \rho\sigma\lambda(F)\rho$

▸**Page 86** line 4 ——— 21 Mar 2010

answer 26 $\rightsquigarrow$ answer 36

Page 87 line 2 of answer 40 ——— 18 Oct 2006

7.1–(00) $\rightsquigarrow$ 7.1.3–(36)

Page 88 new sentence for the end of answer 45 ——— 26 Apr 2007

[See the addendum on page 105.]

▸**Page 90** near the top ——— 23 Oct 2010

line 5: path length $\rightsquigarrow$ level of a leaf
line 6: hit a leaf $\rightsquigarrow$ hit the leftmost leaf
line 10: to a leaf $\rightsquigarrow$ to the leftmost left

Page 92 line 10 ______ 18 Jul 2010

If a_k ... to I5. $\rightsquigarrow$ If $k = 0$, go to I5; if $a_k =$ ')', go to I3; if $a_k =$ '(', go to I4.

▸**Page 98** line 3 (not line 2) of step F4 in answer 99 ______ 26 Mar 2010

$v_q \leftarrow k \oplus 1 \rightsquigarrow v_q \leftarrow k$

Page 100 line −2 of answer 103 ______ 18 Jul 2010

7–00 $\rightsquigarrow$ 7–137

Page 100 line −2 ______ 03 Apr 2007

Kronecker product $\rightsquigarrow$ direct product

Page 101 in answer 105(e) ______ 05 Apr 2007

line 1: $d' \rightsquigarrow d$ (twice)
line 2: $e_{ij} \rightsquigarrow a_{ij}$

Page 101 new paragraphs for answer 105 ______ 05 Apr 2007

(h) When G' is regular, we can make $S^- A' S$ a diagonal matrix with entries $d' - \alpha'_j$, while simultaneously $S^- J_{n'} S$ is a diagonal matrix with entries $(n', 0, \ldots, 0)$, because $(1, \ldots, 1)^T$ is an eigenvector of both A' and $J_{n'}$. Thus, by the formula of answer 7–96(c), the aspects turn out to be $\{d+(d'-\alpha'_j n'[j=0])(d''-\alpha''_k)+(d'-\alpha'_j)(d''-\alpha''_k-n''[k=0]) \mid 0 \le j < n', 0 \le k < n''\}$, where $d = d'(n'' - d'') + (n' - d')d''$.

(i) A similar argument yields the scaled aspects $\{n''\alpha'_j \mid 0 \le j < n'\}$ of G', together with n' copies of shifted aspects $\{d'n'' + \alpha''_k \mid 1 \le k < n''\}$ of G''.

Page 102 notational changes in answer 108 ______ 27 Mar 2007

line 7: $T_n \rightsquigarrow K_n^{\rightarrow}$
line 10: $T_3 \rightsquigarrow K_3^{\rightarrow}$
line 13: $T_3 \rightsquigarrow K_3^{\rightarrow}$
line 22: $Q_n \rightsquigarrow P_n^{\rightarrow}$
line 23: $O_n \rightsquigarrow C_n^{\rightarrow}$

Page 103 more notational changes ______ 27 Mar 2007

line 4: $O_m \times O_n \rightsquigarrow C_m^{\rightarrow} \mathbin{\square} C_n^{\rightarrow}$

▸**Page 104** line 3 of answer 119 ______ 26 Aug 2006

7.2.1.1–5 $\rightsquigarrow$ 7.2.1.1–(5)

▸**Page 104** in answer 122 ______ 09 Jan 2010

line 7: 1640 $\rightsquigarrow$ 1641
line 11: 3365 $\rightsquigarrow$ 3366
line 16: 96,504 $\rightsquigarrow$ 97,221

▸**Page 105** line −4 of answer 122 ______ 09 Jan 2010

(a) 2,747,275 cases; (b) 6,834,708; (c) 741,167,401. $\rightsquigarrow$
(a) 2,735,136 cases; (b) 6,813,760; (c) 739,361,319.

▸**Page 105** line 4 of answer 123 ______ 09 Jan 2010

1361, 2758, and 85054 $\rightsquigarrow$ 1362, 2759, and 85597

▸**Page 105** line 4 of answer 123 ______ 26 Aug 2006

120(a) $\rightsquigarrow$ 122(a)

Page 105 new copy for bottom of page ________ 26 Apr 2007

45, continued. Skarbek has, however, improved Algorithm B to the following Algorithm B*, which generates the trees in reverse order and uses an auxiliary table $c_1 \ldots c_n$:

B1*. [Initialize.] Set $l_k \leftarrow c_k \leftarrow 0$ and $r_k \leftarrow k+1$ for $1 \le k < n$; also set $l_n \leftarrow r_n \leftarrow 0$, and set $r_{n+1} \leftarrow 1$ (for convenience in step B3*).

B2*. [Visit.] Visit the binary tree represented by $l_1 l_2 \ldots l_n$ and $r_1 r_2 \ldots r_n$.

B3*. [Find j.] Set $j \leftarrow 1$. While $r_j = 0$, set $l_j \leftarrow c_j \leftarrow 0$, $r_j \leftarrow j+1$, and $j \leftarrow j+1$. Then terminate the algorithm if $j > n$.

B4*. [Demote r_j.] Set $x \leftarrow r_j$, $r_j \leftarrow r_x$, $r_x \leftarrow 0$, $z \leftarrow c_j$, $c_j \leftarrow x$. If $z > 0$, set $r_z \leftarrow x$; otherwise set $l_j \leftarrow x$. Return to B2*. ▮

If the values of r_1 and c_1 are maintained in registers, this algorithm needs only $4C_n + C_{n-1} + 4(C_{n-1} + C_{n-2} + \cdots + C_0) + 3n - 6 = (67/12 + 73/(24n) + O(n^{-2}))C_n$ mems to generate all C_n trees. [See W. Skarbek, *Fundamenta Informaticæ* **75** (2007), 505–536.]

Page 106 lines 1–4 of answer 4 ________ 26 Apr 2007

this glitch ... this flaw. $\rightsquigarrow$ similarly, one **ילדמג** on line 8 should be **ילמדג**. And the six cases with rightmost letters **גד** appear twice, in lines 3 and 4, while the cases with rightmost **גל** are missing. These glitches are probably typographical and/or scribal errors, not made by Donnolo himself.

Page 106 line 4 of answer 6 ________ 03 Jul 2007

1975 $\rightsquigarrow$ 1963

▸**Page 107** line 2 of answer 8 ________ 12 Jan 2006

$a_1 a_{n-1} \ldots a_n a_3 a_2 \rightsquigarrow a_1 a_4 \ldots a_n a_3 a_2$

Page 107 replacement for the last line of answer 8 ________ 12 Jan 2006

$\pm a_1 \ldots a_{n-1}$ of $\{1, \ldots, n-1\}$ yields n others, $\pm a_1 \ldots a_{n-1} a_n \mp a_1 \ldots a_{n-2} a_n a_{n-1} \pm \cdots$.

▸**Page 107** line 2 of answer 9 ________ 16 Oct 2010

for example, ‘ز’ $\rightsquigarrow$ for example, ‘ر’

▸**Page 108** answer 12 ________ 29 Nov 2009

by 5 $\rightsquigarrow$ by 5 and add 2

▸**Page 109** answer 22 ________ 12 Jan 2006

line 11: exercise 20(c) $\rightsquigarrow$ exercise 21(c)
line 14: exercise 18 $\rightsquigarrow$ exercise 19

Page 110 line −3 of answer 26 ________ 27 Dec 2006

closed form. $\rightsquigarrow$ closed form, although it does satisfy the interesting functional relation $1 + zB(z) = B(z/(1+z))$.

Page 111 new quotations for bottom of page ________ 11 Nov 2010

— HARPO MARX, *The Cocoanuts* (1925)

— MARCEL MARCEAU, *Baptiste* (1946)

Page 112 and following ——— 19 Mar 2006

Miscellaneous changes to the existing index of Volume 4 Fascicle 4 are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.