☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ ERRATA TO VOLUME 4 FASCICLE 3 ☆☆☆☆

This document is a transcript of the notes that I have been making in my personal copy of *The Art of Computer Programming*, Volume 4, Fascicle 3, since it was first printed in 2005.

Four levels of updates — "errors," "amendments," "plans," and "improvements" — appear, indicated by four different typographic conventions:

▸**Page 666** line 1 ______ 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \rightsquigarrow n+1$

Page 666 line 2 ______ 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . .' leading to the date on which I recorded the plan in my files.

Page 666 line 4 ______ 10 Jan 1938

The fourth and final category — indicated by page and line number in smaller, slanted type — consists of minor corrections or improvements that most readers don't want to know about, because they are so trivial. You wouldn't even be seeing these items if you hadn't specifically chosen to print the complete errata list in all its gory details. Are you sure you wanted to do that?

The ultimate, glorious, future editions of Volumes 1–5 are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 4B, Stanford University, Stanford CA 94305-9045, or by email to `taocp@cs.stanford.edu`*. (Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although I'm working full time on*

Volume 4 these days, I will try to reply to all such messages within a year of receipt. Current news about The Art of Computer Programming is posted on

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

and updated regularly.

— Don Knuth, June 2005

For a work of such scope as this,
the first word of the author should be an apology
for what is doubtless the too ambitious effort
of a single writer.

— EDWARD W. BYRN (1900)

GENERATING ALL COMBINATIONS AND PARTITIONS

Last updated 1 December 2010

Most of these corrections have already been made in recent printings.

Important note: The changes below include nearly every difference between the paperback fascicle and the first printing of Volume 4A, published in January 2011. All subsequent changes can be found in the errata list for that volume.

▸**Page ii** in the ISBN numbers ______ 29 Apr 2008

line 19: `1-201-85394-9` $\leadsto$ `0-201-85394-9`
line −4: 1-201-85394-9 $\leadsto$ 0-201-85394-9

Page iv line 20 ______ 28 Oct 2008

I shall happily pay a finder's fee of \$2.56 $\leadsto$ I happily offer a "finder's fee" of \$2.56

Page iv lines 24 and 25 ______ 28 Oct 2008

reward you with immortal glory instead of mere money $\leadsto$ do my best to give you immortal glory

Page iv line −12 ______ 04 Dec 2005

indices $\leadsto$ indexes

Page 0 replacement for lines 3 and 4 ______ 12 Jan 2009

The opening sections of this chapter appear in Volume 4, Fascicle 0, and Volume 4, Fascicle 1, first published in 2008 and 2009.

Page 1 line −8 ______ 21 Apr 2008

The latter representation $\leadsto$ The string representation

▸**Page 2** in the running headline (also on pages 4, 6, 8, ...!) ______ 07 Sep 2005

(F2) $\leadsto$ (F3)

Page 2 line 7 ______ 12 Oct 2008

multicombination, $\leadsto$ *multicombination* of $s+1$ things,

Page 4 line 10 ______ 18 Oct 2006

7.1–00 $\leadsto$ 7.1.3–20

Page 4 line −7 ______ 17 Sep 2005

repeat until $c_j + 1 \neq c_{j+1}$. $\leadsto$ eventually the condition $c_j + 1 \neq c_{j+1}$ will occur.

Page 5 second line of step T4 ______ 13 Mar 2008

repeat this step until $x \neq c_{j+1}$ $\leadsto$ repeat step T4

▸**Page 7** in (23) ______ 31 Jul 2006

$w_1 \ge w_0$; $\leadsto$ $w_1 \ge w_0 \ge 0$;

Page 8 lines -13 and -12 ———— 24 Mar 2006

discovered by D. T. Tang $\leadsto$ discovered by J. E. Miller in her Ph.D. thesis (Columbia University, 1971), then independently by D. T. Tang

Page 8 near the bottom ———— 23 Oct 2006

lines -3 and -1: increasing $\leadsto$ nondecreasing
line -2: *decreasing* $\leadsto$ non*increasing*

Page 13 in step C3 ———— 10 Aug 2009

If $w_j = 0$, set $w_j \leftarrow 1$, $j \leftarrow j+1$, and repeat until $w_j = 1$. $\leadsto$ While $w_j = 0$, set $w_j \leftarrow 1$ and $j \leftarrow j+1$.

Page 13 in steps C4 and C6 ———— 21 Apr 2008

$r = j > 1 \leadsto r = j$ and $j > 1$

▸**Page 17** (and throughout the fascicle!) ———— 04 Apr 2008

Fig. 26 $\leadsto$ **Fig. 46** [and all figure numbers need to increased by 20, except that there are *two* instances of Fig. 35! There are separate errata entries for the Fig. 35 that becomes Fig. 56.]

Page 19 line -2 ———— 07 Jun 2010

we have $\leadsto$ with $\kappa_t 0 = 0$, we have

Page 20 line -2 ———— 07 Aug 2005

in cross order $\leadsto$ in increasing cross order

Page 22 line -5 ———— 30 May 2010

In other words, $\leadsto$ *Proof.* In other words,

Page 27 exercise 20 ———— 30 May 2010

Find $\leadsto$ Devise

Page 27 exercise 21 ———— 24 Mar 2006

Prove $\leadsto$ (Joan E. Miller, 1971.) Prove

▸**Page 35** in the running headline ———— 25 Oct 2005

PARTITIONS $\leadsto$ COMBINATIONS

▸**Page 35** line 3 of exercise 34 ———— 07 Mar 2006

11001000, 11001001 $\leadsto$ 11001000, 11001010, 11001001

Page 35 new exercise ———— 09 Aug 2006

▸**111.** [*M26*] (P. Erdős, C. Ko, and R. Rado.) Suppose A is a set of r-combinations of an n-set, with $\alpha \cap \beta \neq \emptyset$ whenever $\alpha, \beta \in A$. Show that $|A| \leq \binom{n-1}{r-1}$, if $r \leq n/2$. *Hint:* Consider $\partial^{n-2r} B$, where B is the set of complements of A.

Page 36 line 4 ———— 21 Aug 2007

(see Table 1). All twelve of Stanley's $\leadsto$
(see Table 1), based on a series of lectures by Gian-Carlo Rota. All twelve of these

Page 37 line 12 — 22 Sep 2008

into disjoint subsets $\rightsquigarrow$ into nonempty, disjoint subsets

Page 37 bottom line — 24 Oct 2005

52]: $\rightsquigarrow$ 52], and pad the array with 1s as suggested by A. Zoghbi and I. Stojmenović [*International Journal of Computer Math.* **70** (1998), 319–332]:

Page 38 line 3 — 29 May 2006

$n \geq 1$. $\rightsquigarrow$ $n \geq 1$. The value of a_0 is also set to zero.

Page 38 replacement for step P1 — 24 Oct 2005

P1. [Initialize.] Set $a_m \leftarrow 1$ for $n \geq m > 1$. Then set $m \leftarrow 1$ and $a_0 \leftarrow 0$.

Page 38 replacement for step P4 — 24 Oct 2005

P4. [Change 2 to 1+1.] Set $a_q \leftarrow 1$, $q \leftarrow q - 1$, $m \leftarrow m + 1$, and return to P3. (At this point we have $a_k = 1$ for $q < k \leq n$.)

Page 38 line 3 of Algorithm H — 30 May 2010

$n \geq m \geq 2$. $\rightsquigarrow$ $n \geq m \geq 2$. A sentinel value is stored in a_{m+1}.

Page 38 better way to state step H4 — 04 Aug 2009

H4. [Find j.] Set $j \leftarrow 3$ and $s \leftarrow a_1 + a_2 - 1$. Then, while $a_j \geq a_1 - 1$, set $s \leftarrow s + a_j$ and $j \leftarrow j + 1$. (Now $s = a_1 + \cdots + a_{j-1} - 1$ and $a_j < a_1 - 1$.)

Page 39 line −16 — 22 Sep 2008

n-multicombinations of N $\rightsquigarrow$ n-multicombinations of N things

Page 40 line 5 before (11) — 14 Jun 2010

by transposing the rows and columns of the corresponding Ferrers diagram. $\rightsquigarrow$ by transposing its Ferrers diagram — that is, by reflecting the diagram about the main diagonal.

Page 40 line 6 after (11) — 22 Sep 2008

$(a_1 \ldots a_m)^T$ $\rightsquigarrow$ $(a_1 \ldots a_m)^T$, if $m < n$

Page 40 line 3 before (13) — 25 Aug 2007

representation. $\rightsquigarrow$ *representation* [see S. Comét, *Numer. Math.* **1** (1959), 90–109].

Page 41 line 1 — 22 Sep 2008

a path $\rightsquigarrow$ a path of length $2n$

Page 41 lines 10 and 11 — 14 Jun 2010

When the partition has t different parts, $\rightsquigarrow$ When the number of distinct parts of a partition is equal to t,

Page 41 line 18 — 23 Aug 2010

Philipp $\rightsquigarrow$ Philippe

Page 42 line 1 ——— 26 Sep 2010

identity of Jacobi $\leadsto$ identity that Jacobi published in 1829

Page 42 the line after (21) ——— 14 Jun 2010

setting $z = e^{-t}$ $\leadsto$ setting $z = e^{-t}$ with $t > 0$

Page 42 in (23) ——— 14 Jun 2010

$$\frac{p(n)}{1-e^{-t}} \leadsto \frac{p(n)}{1-e^{-t}} = \sum_{k=n}^{\infty} p(n)e^{(n-k)t}$$

Page 43 lines 10 and 11 ——— 14 Jun 2010

$0 \le \Re\theta \le 1$ ... rectangle; $\leadsto$

▸**Page 43** line 15 ——— 11 Jun 2010

Theorem 10.6.2 $\leadsto$ Theorem 10.6e

Page 43 replacement for lines -11 through -2 ——— 27 Sep 2010

$$f(y) \;=\; e^{-\frac{3}{2}t(y+\frac{1}{6})^2+\pi iy}; \tag{27}$$

and this function f qualifies for Poisson's summation formula under both of the criteria (i) and (ii) stated above. Therefore we try to integrate $e^{-2\pi miy}f(y)$, and that integral turns out to be easy for all m (see exercise 27):

$$\int_{-\infty}^{\infty} e^{-a(y+b)^2+2ciy}\,dy \;=\; \sqrt{\frac{\pi}{a}}\,e^{-c^2/a-2bci} \qquad \text{when } a > 0. \tag{28}$$

Plugging in to (25), with $\theta = 0$, $a = \frac{3}{2}t$, $b = \frac{1}{6}$, and $c = (\frac{1}{2} - m)\pi$, yields

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{m=-\infty}^{\infty} g(m), \qquad g(m) = \sqrt{\frac{2\pi}{3t}}\,e^{-2(m-\frac{1}{2})^2\pi^2/(3t)+\frac{1-2m}{6}\pi i}. \tag{29}$$

These terms combine and cancel beautifully, as shown in exercise 27, giving

$$\frac{e^{-t/24}}{P(e^{-t})} \;=\; \sqrt{\frac{2\pi}{t}}\sum_{n=-\infty}^{\infty}(-1)^n e^{-6\pi^2(n+\frac{1}{6})^2/t} \;=\; \sqrt{\frac{2\pi}{t}}\,\frac{e^{-\zeta(2)/t}}{P(e^{-4\pi^2/t})}. \tag{30}$$

Page 45 line 3 before (40) ——— 18 Apr 2006

by considering $\leadsto$ by transposing

Page 46 line 2 below Table 2 ——— 24 Mar 2008

Erdős $\leadsto$ Erdös [to agree with the English spelling in the paper cited]

▸**Page 47** line 2 before (47) ——— 22 Sep 2008

$\alpha^m = n^{-1/2}e^{-Cx}$ $\leadsto$ $\alpha^m = n^{-1/2}e^{-Cx} + O(n^{-1})$

▸**Page 48** line 2 after the top illustration ——— 22 Sep 2008

$O(\sqrt{n}\,)$ $\leadsto$ $O(\sqrt{n}\log\log\log n)$

Page 48 the line after (50) ——— 14 Jun 2010

m is reasonably large $\leadsto$ m and n are reasonably large

Page 49 two lines before (52) ——— 14 Jun 2010

in steps P2 and P6; $\leadsto$ in step P2 or when we'll soon test $n \le x$ in P6;

▸**Page 49** line −1 ——— 14 May 2009

$4 - 3C/\sqrt{n} \leadsto 3 + C/\sqrt{n}$

Page 50 line 10 ——— 22 Sep 2008

$(a_1 \ldots a_m)^T \leadsto (a_1 \ldots a_m)^T$ and $m < n$

Page 51 line 4 after Table 3 ——— 30 May 2010

$a_1a_2 \ldots a_n \leadsto a_1a_2 \ldots$

▸**Page 53** bottom line ——— 08 Dec 2006

l_2, l_4, l_5, $l_7 \leadsto l_2$, l_4, l_5, l_6, l_7

Page 54 new wording for exercise 8 ——— 14 Jun 2010

8. [*15*] When $(p_1 \ldots p_t, q_1 \ldots q_t)$ yields the rim representation of a partition $a_1a_2\ldots$ as in (15) and (16), what's the rim representation of the conjugate partition $(a_1a_2\ldots)^T$?

▸**Page 54** new wording for exercise 13 ——— 26 Sep 2010

▸**13.** [*M23*] (F. Franklin, 1882.) Find a one-to-one correspondence $\alpha \leftrightarrow \beta$ between partitions of n such that α has exactly k parts repeated more than once if and only if β has exactly k even parts. (For example, the partition 64421111 has two repeated parts $\{4, 1\}$ and three even parts $\{6, 4, 2\}$. The case $k = 0$ corresponds to Euler's result.)

Page 54 new rating for exercise 14 ——— 14 Jun 2010

[*M28*] $\leadsto$ [*M33*]

Page 54 new wording for exercise 16 ——— 22 Sep 2008

Find ... and sum $\leadsto$ Find a formula for $\sum_{m,n} p(k,m,n)w^m z^n$, where $p(k,m,n)$ is the number of partitions of n that have m parts and trace k. Sum

Page 55 new sentence for end of exercise 17(b) ——— 14 Jun 2010

For example, the answer when $t = 2$ is $(1+v)(1+vz)z^2/((1-z)(1-z^2))$.

Page 55 replacement for the last line of exercise 18 ——— 14 Jun 2010

$$\sum_{\substack{a_1\ge\cdots\ge a_r>0,\ r\ge0\\ b_1>\cdots>b_s>0,\ s\ge0}} u^{r+s}v^s z^{a_1+\cdots+a_r+b_1+\cdots+b_s} \;=\; \sum_{\substack{c_1\ge\cdots\ge c_t>0,\ t\ge0\\ d_1,\ldots,d_t\in\{0,1\}}} u^t v^{d_1+\cdots+d_t} z^{c_1+\cdots+c_t+(t-1)d_1+\cdots+d_{t-1}}.$$

Page 54 new rating for exercise 19 ——— 14 Jun 2010

[*M21*] $\leadsto$ [*M22*]

Page 55 line 1 of exercise 21 ——— 22 Sep 2008

partitions into $\leadsto$ partitions of n into

Page 56 in exercise 27 ——— 27 Sep 2010

[*HM23*] Evaluate (29) $\leadsto$ [*HM21*] Prove (28)

Page 57 new wording for exercise 38 ——— 14 Jun 2010

What is ... largest part l? ⇝ Given positive integers l and m, what generating function enumerates partitions that have exactly m parts, and largest part l? (See Eq. (51).)

Page 57 new rating for exercise 40 ——— 28 Sep 2010

[*M22*] ⇝ [*M20*]

Page 57 line 1 of exercise 40 ——— 28 Sep 2010

What is the generating function for partitions ⇝ Continuing exercise 38, what is the generating function for the number of partitions
[Note: Exercises 39 and 40 will become exercises 40 and 39 in Volume 4A.]

Page 57 new rating for exercise 43 ——— 18 Jun 2010

[*M21*] ⇝ [*M18*]

Page 57 in exercise 47 ——— 23 Feb 2007

line 1 of step N4: 1, 2, ..., n ⇝ 1, 2, ...,

line -1 (the displayed equation): $\sum_{j=1}^{m}$ ⇝ $\sum_{j=1}^{\infty}$

Page 58 line 3 of exercise 50 ——— 22 Sep 2008

for all m ⇝ for all $m \ge 0$.

Page 58 line 1 of exercise 54 ——— 14 Jun 2010

The partition $\ldots b_1 b_2 \ldots,$ ⇝ Let $\alpha = a_1 a_2 \ldots$ and $\beta = b_1 b_2 \ldots$ be partitions of n. We say that α *majorizes* β,

▸**Page 58** line 9 of exercise 55 ——— 11 Jun 2010

$n_2 > n_1 \ge 0$ ⇝ $n_2 > n_1 \ge 0$ and $n_2 > 2$

Page 59 the rating of exercise 56 ——— 21 Jan 2007

[*M27*] ⇝ [*M32*]

Page 60 line 2 of exercise 57 ——— 14 Jun 2010

Then ⇝ By permuting rows and columns we can assume that $r_1 \ge r_2 \ge \cdots$ and $c_1 \ge c_2 \ge \cdots$. Then

▸**Page 60** line 2 of exercise 59 ——— 26 Oct 2006

same as conjugate ⇝ same as the conjugate

Page 61 new exercise ——— 11 Sep 2009

72. [*M30*] How many partitions of n have no predecessor in Bulgarian solitaire?

[The former exercise 72 now becomes number 73.]

Page 62 line 18 ——— 30 Oct 2005

in which we have ⇝ of nonnegative integers in which we have

Page 63 an explanatory sentence to follow (9) ———— 09 Feb 2008

(In other words, we begin with either $A_{(m-1)n}(m-1)$ or $A'_{(m-1)n}(m-1)$ and then use either $A^R_{mn}j$ or $A_{mn}j$, alternately, as j decreases from $m-1$ to 0.)

Page 64 line −9 ———— 09 Oct 2010

Annals of Math. $\leadsto$ *Annals of Math.* (2)

Page 67 line 2 ———— 30 Nov 2008

$\max(\Re z, \Im z)$ becomes $\max\bigl(|\Re z|, |\Im z|\bigr)$

▸**Page 67** lines 6 and 7 ———— 08 Nov 2010

and if the sums $\sum_{t\in B\setminus A} f(t)$ and $\sum_{t\in C\setminus A} \hat f(t)$ are both small, then $\sum_{t\in C} \hat f(t)$ is a good approximation to $\sum_{t\in B} f(t)$. $\leadsto$
and if the sums $\sum_{t\in B} |f(t)|$ and $\sum_{t\in C} |\hat f(t)|$ are both small, then $\sum_{t\in A\cup C} \hat f(t)$ is a good approximation to $\sum_{t\in A\cup B} f(t)$.

▸**Page 67** line 14 ———— 23 Aug 2005

$O\Bigl(\frac{e^{n-n^{-2\epsilon}/2}}{n^n}\Bigr) \leadsto O\Bigl(\frac{e^{n-n^{2\epsilon}/2}}{n^n}\Bigr)$

Page 67 line −3 ———— 30 Nov 2008

$O(n^{9\epsilon-3/2}) \leadsto O(n^{9\epsilon-2})$

Page 68 on the line following (23) ———— 17 Sep 2005

saddle point now occurs $\leadsto$ saddle point for the new integrand occurs

Page 69 line −2 ———— 17 Jun 2010

b_0, b_1, b_2, b_3 $\leadsto$ 1, b_1/n, b_2/n^2, b_3/n^3

▸**Page 72** line −20 ———— 14 May 2009

By comparing (47) $\leadsto$ By comparing (46)

Page 74 line 18 ———— 14 May 2009

each k-block partition $\leadsto$ each set partition that has k blocks

Page 74 line −8 ———— 18 Oct 2010

different multisets $\leadsto$ types of multisets

Page 75 line 22 ———— 08 Feb 2008

simple $\leadsto$ fairly simple

▸**Page 75** replacement for (57) ———— 18 Jan 2007

j	0	1	2	3	4	5	6	7	8	9	10	11	
c_j	1	2	1	2	1	2	1	2	1	2	2	2	
u_j	9	9	6	8	4	6	2	6	1	5	4	1	
v_j	3	1	2	2	2	0	1	1	1	1	3	1	
	$f_0=0$		$f_1=2$		$f_2=4$		$f_3=6$		$f_4=8$		$f_5=10$	$f_6=11$	$f_7=12$

(57)

▸**Page 75** line −2 — 18 Jan 2007

$f_0 f_1 \ldots f_n$, $c_0 c_1 \ldots c_n$, $u_0 u_1 \ldots u_n$, and $v_0 v_1 \ldots v_n$ $\rightsquigarrow$
$f_0 f_1 \ldots f_n$, $c_0 c_1 \ldots c_{mn}$, $u_0 u_1 \ldots u_{mn}$, and $v_0 v_1 \ldots v_{mn}$

Page 76 restatement of step M2 — 09 Feb 2008

Set $j \leftarrow a$ and ... until $j = b$. $\rightsquigarrow$ Set $j \leftarrow a$, $k \leftarrow b$, and $x \leftarrow 0$. Then while $j < b$ do the following: Set $u_k \leftarrow u_j - v_j$. If $u_k = 0$, just set $x \leftarrow 1$ and $j \leftarrow j+1$. Otherwise if $x = 0$, set $c_k \leftarrow c_j$, $v_k \leftarrow \min(v_j, u_k)$, $x \leftarrow [u_k < v_j]$, $k \leftarrow k+1$, $j \leftarrow j+1$. Otherwise set $c_k \leftarrow c_j$, $v_k \leftarrow u_k$, $k \leftarrow k+1$, $j \leftarrow j+1$. (Notice that $x = [v$ has changed].)

Page 76 line 25 — 18 Jan 2007

every component. $\rightsquigarrow$ every component. (See exercise 68.)

▸**Page 80** line −3 — 06 Jun 2008

Fig. 35. $\rightsquigarrow$ Fig. 56

▸**Page 81** caption to the illustration — 06 Jun 2008

Fig. 35. $\rightsquigarrow$ **Fig. 56**

▸**Page 81** line 3 after the illustration — 06 Jun 2008

Fig. 35. $\rightsquigarrow$ Fig. 56

Page 81 line 7 after the illustration — 17 Jun 2010

Ferrers diagram $\rightsquigarrow$ tableau shape

▸**Page 81** in exercise 29 — 06 Jun 2010

line 8: $a_1 \geq \cdots \geq a_m$ $\rightsquigarrow$ $a_1 \geq \cdots \geq a_m \geq 0$
line 9 (the displayed formula): $y^{a_j+m-j+t}$ $\rightsquigarrow$ $y^{a_j+j-m+t}$

▸**Page 82** bottom line of exercise 29 — 14 May 2010

multisets $\{a_1{+}m, a_2{+}m{-}1, \ldots, a_m{+}1\}$ and $\{a'_1{+}m, a'_2{+}m{-}1, \ldots, a'_m{+}1\}$ $\rightsquigarrow$ multisets $\{a_1{+}1, a_2{+}2, \ldots, a_m{+}m\}$ and $\{a'_1{+}1, a'_2{+}2, \ldots, a'_m{+}m\}$

▸**Page 85** line 3 of exercise 63 — 01 Jul 2006

$(1{+}p_n(1{-}z)$ $\rightsquigarrow$ $(1{+}p_n(z{-}1)$

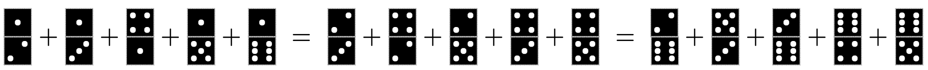
Page 85 line 1 of exercise 68 — 18 Jan 2007

[*20*] How large can the stack get in Algorithm M, when it $\rightsquigarrow$ [*21*] How large can variables l and b get in Algorithm M, when that algorithm

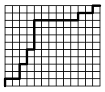
Page 85 line 1 of exercise 69 — 09 Feb 2008

[*21*] $\rightsquigarrow$ [*22*]

▸**Page 86** replacement for dominoes in exercise 82 — 17 Sep 2005



Page 87 replacement for second line of answer 2 ——— 10 Mar 2010

result is . Conversely, we can easily "read off" the representations a_i, b_i, c_i, d_i, p_i, or q_i of (2)–(11) from any given path from $(0,0)$ to (s,t).

Page 87 in answer 3 ——— 30 Jan 2010

[change the step numbers F1, F2, F3, F4, F5 respectively to N1, N2, N3, N4, N5]

▸**Page 87** last line of answer 5 ——— 31 Jul 2006

$(\delta_t,\ldots,\delta_1) \rightsquigarrow (\delta_t,\ldots,\delta_0)$

▸**Page 87** last line of answer 6 ——— 30 May 2010

$\binom{n-2}{t-1}+\binom{n-2}{t-2}+\cdots+\binom{n-t-1}{0} \rightsquigarrow \binom{n-2}{t-1}+\binom{n-3}{t-2}+\cdots+\binom{n-t-1}{0}$

Page 87 step S4 in answer 7 ——— 10 Aug 2009

set $b_{j-1} \leftarrow b_j - 1$, $j \leftarrow j-1$, and repeat until $j=1$. $\rightsquigarrow$ set $b_{j-1} \leftarrow b_j - 1$ and $j \leftarrow j-1$.

▸**Page 89** line 5 of answer 12(b) ——— 30 May 2010

10 $\rightsquigarrow$ **A10**

Page 89 step V4 in answer 9 ——— 10 Aug 2009

If $a_{k(q+r)}=1$, set $a_{k(q+r)} \leftarrow 0$, $k \leftarrow k+1$, and repeat until $a_{k(q+r)}=0$. $\rightsquigarrow$

▸**Page 90** replacement for answer 19 ——— 18 Mar 2006

19. It is the binary representation of post(1000000), where $\mathrm{post}(2^{k+1}-1)=2^k$, and $\mathrm{post}(n)=2^k+\mathrm{post}(n-2^k+1)$ if $2^k \le n < 2^{k+1}-1$, for $k \ge 0$. So it is 11110100001001000100.

Page 90 addendum to answer 19 ——— 02 Oct 2006

[Incidentally, the left-child/right-sibling representation of T_∞ is the sideways heap.]

Page 90 line 1 of answer 21 ——— 11 Oct 2008

[See $\rightsquigarrow$ [Page 40 of Miller's thesis; see also

▸**Page 91** line 13 of answer 25 ——— 26 Apr 2006

$\{b_s,\ldots,b_1,c_t\ldots,c_1\} \rightsquigarrow \{b_s,\ldots,b_1,c_t,\ldots,c_1\}$

▸**Page 92** line 6 of answer 29 ——— 06 Jun 2010

$1 \le j \le t \rightsquigarrow 1 \le j \le s$

▸**Page 92** line −2 of answer 30 ——— 30 May 2010

$\widehat{C}_{st}$ is $A_{\lfloor 2^s-1/3\rfloor}(s,t) \rightsquigarrow \widehat{C}_{st}$ is $A_{\lfloor 2^s-1/3\rfloor}(s,t)^R$

Page 95 addendum to answer 43 ——— 08 Dec 2006

(Incidentally, $s(x)=f(x)\oplus 1$ and $p(x)=f(x\oplus 1)$, where $f(x)=(x \mathbin{\dot-} 1)+((x\,\&\,1)\ll 1)$.)

▸**Page 97** line 1 of answer 55 ——— 06 Sep 2005

[Solution $\rightsquigarrow$ (a) [Solution

▸**Page 97** line 4 of answer 55 ——— 03 Dec 2005

$t \le i \le n \rightsquigarrow t \le j \le n$

Page 98 line 5 — 26 Jan 2010

576.] $\leadsto$ 576; see also A. Williams, *SODA* **20** (2009), 987–996, for a significant generalization by which the permutations of an arbitrary multiset can be generated looplessly by prefix rotations.]

▸**Page 98** line 4 of answer 60 — 07 Feb 2008

$x = t \leadsto x \leftarrow t$.

Page 98 in step Q2 of answer 60 — 10 Aug 2009

$j \leftarrow j+1$, and repeat until $x \le m_j$. $\leadsto$ and $j \leftarrow j+1$.

▸**Page 99** line −6 of answer 62 — 19 May 2010

$r_{i+1} + \cdots + r_m - c_1 - \cdots - c_{j-1}, c_1 + \cdots + c_j - r_i - \cdots - r_m \leadsto$
$r_1 + \cdots + r_i - c_{j+1} - \cdots - c_n, c_j + \cdots + c_n - r_1 - \cdots - r_{i-1}$

Page 99 line −5 of answer 62 — 06 Sep 2005

[This line, the beginning of part (e), should be indented.]

Page 101 line 1 of answer 70 — 30 May 2010

From the identity $\leadsto$ Using the facts that $t \ge 3$ implies

▸**Page 101** line 3 of answer 70 — 06 Jun 2010

$\binom{2t-2}{t}\frac{1}{t} \leadsto \binom{2t-2}{t}\frac{1}{t-1}$

▸**Page 102** line −4 of answer 77 — 30 May 2010

$\binom{n_v+1}{v} \leadsto \binom{n_v+1}{v+1}$

▸**Page 106** last line of answer 97 — 09 Jun 2010

$13300 \leftrightarrow 14400 \leadsto 13300 \leftrightarrow 14310$

▸**Page 108** line 2 of answer 107 — 09 Aug 2006

3-combinations $\leadsto$ 2-combinations

Page 109 new answer — 09 Aug 2006

111. $\partial^{n-2r}B$ is the set of all r-subsets of B; these subsets must not be in A. If $|A| = |B| = \binom{x}{n-r}$ for some real $x > n-1$, we would have $\binom{n}{r} \ge |A| + |\partial^{n-2r}B| \ge \binom{x}{n-r} + \binom{x}{r} > \binom{n-1}{n-r} + \binom{n-1}{r} = \binom{n}{r}$, by exercise 80. [See *Quart. J. Math. Oxford* **12** (1961), 313–320.]

▸**Page 110** in the running headline — 27 Aug 2005

7.2.1.3 $\leadsto$ 7.2.1.4

Page 110 line 1 of answer 4 — 11 Jun 2010

We must $\leadsto$ Assume that $1 \le r \le n$. We must

Page 110 simplification of step C4 in answer 5 — 18 Jun 2010

$c_2 \leftarrow c2 + 1$. Then $\ldots$ $l_1 \leftarrow 2$. $\leadsto$ $c_2 \leftarrow c_2 + 1$, and $l_{[c_1>0]} \leftarrow 2$. If $k \ne 2$, also set $l_2 \leftarrow k$. Return to C2.

Page 110 improved answer 6 ———— 27 Sep 2010

6. Assume that each partition is followed by 0. Set $j \leftarrow 0$, $k \leftarrow a_1$, and $b_{k+1} \leftarrow 0$. Then, while $k > 0$, set $j \leftarrow j + 1$ and, while $k > a_{j+1}$, set $b_k \leftarrow j$ and $k \leftarrow k - 1$. (We have used (11) in the dual form $a_j - a_{j+1} = d_j$, where $d_1 \ldots d_n$ is the part-count representation of $b_1 b_2 \ldots$. This algorithm essentially walks along the rim of the Ferrers diagram; so its running time is roughly proportional to $a_1 + b_1$, the number of parts in the output plus the number of parts in the input.)

Page 110 last line of answer 8 ———— 14 Jun 2010

$b_1 b_2 \ldots \leadsto (a_1 a_2 \ldots)^T$

▸**Page 111** line 3 of answer 11 ———— 22 Sep 2008

infinite series $\prod_{k\ge1}(1 + z^k + z^{2k})(1 + z^{2k} + z^{4k})(1 + z^{5k} + z^{10k}) \leadsto$
infinite product $\prod_{k\ge0}\prod_{r\in\{10^k,2\cdot10^k,5\cdot10^k\}}(1 + z^r + z^{2r})$

Page 111 replacement for answer 13 ———— 26 Sep 2010

13. Map the partition $c_1{\cdot}1 + c_2{\cdot}2 + c_3{\cdot}3 + \cdots$ into $r_1{\cdot}1 + \lfloor c_1/2\rfloor{\cdot}2 + r_3{\cdot}3 + \lfloor c_2/2\rfloor{\cdot}4 + r_5{\cdot}5 + \lfloor c_3/2\rfloor{\cdot}6 + \cdots$, where $r_m = (c_m \bmod 2) + 2(c_{2m} \bmod 2) + 4(c_{4m} \bmod 2) + 8(c_{8m} \bmod 2) + \cdots$; $433222211 \mapsto 64421111 \mapsto 8332211$. [*Johns Hopkins Univ. Circular* **2** (1882), 72.]

▸**Page 111** line 2 of answer 14 ———— 22 Sep 2008

odd permutation are centered and divided $\leadsto$ odd parts are centered and the partition is divided

Page 111 replacement for second paragraph of answer 14 ———— 18 Jun 2008

In general, when the "Durfee rectangle" (shown in the diagram) has t rows, suppose there are $a_1, \ldots, a_t$ extra dots at the right and $b_t, \ldots, b_1, \ldots, b_t$ extra dots below, where $a_1 \ge \cdots \ge a_t \ge 0$ and $b_1 \ge \cdots \ge b_t \ge 0$. Then the distinct parts obtained are $2t - 1 + a_1 + b_1$, $2t - 2 + a_1 + b_2$, $\ldots$, $2 + a_{t-1} + b_t$, $1 + a_t + b_t$, and (if it's nonzero) $0 + a_t$. Conversely, any partition with $2t$ distinct nonnegative parts can uniquely be written in this form.

Page 112 line 10 of answer 17 ———— 23 Oct 2006

"Partition bijections, a survey," to appear in *The Ramanujan Journal*. $\leadsto$ *The Ramanujan Journal* **12** (2006), 5–75.

▸**Page 112** line −3 of answer 17 ———— 18 Jun 2010

$r = 0 \leadsto s = 0$

▸**Page 112** line −3 of answer 17 ———— 31 Jul 2006

$b_1 < t). \leadsto b_1 < t.$

Page 113 replacement for first paragraph of answer 23 ———— 18 Jun 2010

23. (Solution by Marc van Leeuwen.) Divide (19) by $1 - v$, to get

$$\prod_{k=1}^{\infty}(1 - u^k v^{k-1})(1 - u^k v^k)(1 - u^k v^{k+1}) = \sum_{n=0}^{\infty}(-1)^n u^{\binom{n+1}{2}}\left(\frac{v^{\binom{n}{2}} - v^{\binom{n+2}{2}}}{1 - v}\right) = \sum_{n=0}^{\infty}(-1)^n u^{\binom{n+1}{2}}\sum_{k=0}^{2n} v^k;$$

now set $u = z$ and $v = 1$.

▸**Page 114** line 2 of answer 24 ——— 22 Sep 2008

summed over all integers j and k $\rightsquigarrow$ summed over all integers j and all nonnegative integers k

▸**Page 114** line 7 of answer 24 ——— 22 Sep 2008

$/P(z^5)$ $\rightsquigarrow$ $/P(z^5)^5$

▸**Page 114** line 4 of answer 26 ——— 22 Sep 2008

$e^{-\pi^2 m^2 n^2}$ $\rightsquigarrow$ $e^{-\pi^2 m^2 n}$

Page 114 Replacement for answer 27 ——— 27 Sep 2010

27. First, $\int_{-\infty}^{\infty} e^{-a(y+b)^2+2ciy}\,dy = e^{-c^2/a-2bci}\int_{-\infty}^{\infty} e^{-au^2}\,du$, by the substitution $u = y + b - ci/a$. And $\int_{-\infty}^{\infty} e^{-au^2}\,du = \int_0^{\infty} e^{-t}\,dt/\sqrt{at} = \Gamma(\frac12)/\sqrt{a} = \sqrt{\pi/a}$, by the substitution $t = au^2$ and exercises 1.2.5–20, 1.2.6–43.

Now (30) follows from (29) because we have, for all integers m,

$$g(3m+1) + g(-3m) = \sqrt{\frac{2\pi}{t}}(-1)^m e^{-6\pi^2(m+\frac16)^2/t};\quad g(3m+2) + g(-3m-1) = 0.$$

[See M. I. Knopp, *Modular Functions in Analytic Number Theory* (1970), Chapter 3.]

▸**Page 115** answer 28 ——— 22 Sep 2008

line -2: If $n =$ $\rightsquigarrow$ If $k =$
line -1: $2^{-t}(1+(-1)^{[e_1=1]}/p_1)\ldots(1+(-1)^{[e_t=1]}/p_t)$ $\rightsquigarrow$
$2^{-t}(1+(-1)^{[e_1>1]}/p_1)\ldots(1+(-1)^{[e_t>1]}/p_t)$

Page 115 line 7 of answer 31 ——— 18 Apr 2010

f_{pq} $\rightsquigarrow$ $f_{p,q}$ (twice)

▸**Page 115** line 7 of answer 31 ——— 22 Sep 2008

less than m/q $\rightsquigarrow$ less than $\lfloor m/q \rfloor$

▸**Page 115** line 2 of answer 33 ——— 22 Sep 2008

$m = \sqrt{n}$ $\rightsquigarrow$ $m = \lfloor\sqrt{n}\rfloor$

▸**Page 116** last line of answer 37 ——— 26 Aug 2006

de Scienze Economiche e Commerciale $\rightsquigarrow$ *di Scienze Economiche e Commerciali*

Page 116 new answer 40 ——— 28 Sep 2010

40. $[x^m]\,(1+zx)(1+z^2x)\ldots(1+z^{l-1}x) = z^{m(m+1)/2}\binom{l-1}{m}_z$, by exercise 1.2.6–58; this is $(z-z^l)(z^2-z^l)\ldots(z^m-z^l)/((1-z)(1-z^2)\ldots(1-z^m))$. The answer also follows from Theorem C: Replacing $a_1\ldots a_m$ by $(a_1-m)\ldots(a_m-1)$ gives an equivalent partition of $n-m(m+1)/2$ into at most m parts, not exceeding $l-1-m$.

Page 117 new answer 43 ——— 18 Jun 2010

43. $p(n-k(k-1)/2)$. (Change $a_1a_2\ldots a_k$ to $(a_1-k+1)(a_2-k+2)\ldots a_k$ to get an equivalent partition of $n-k(k-1)/2$.)

Page 118 part (e) of answer 50 ——— 30 May 2010

line 1: The $\rightsquigarrow$ Continuing the previous exercise and its answer, the
lines 2 and 3: $(1-z)$, where $F_k(z)$ is defined in the previous exercise. $\rightsquigarrow$ $(1-z)$.

▸**Page 118** last line of answer 53 ——— 07 Feb 2007

20 8 8 8 6 5 4 3 3 3 3 3 2 1^{19} $\leadsto$ 20 8 8 8 8 6 5 4 3 3 3 3 3 2 1^{19}

▸**Page 118** part (b) of answer 54 ——— 05 Nov 2005

$n - kl \leadsto n + kl$ (three changes)

▸**Page 118** line 3 of answer 54 ——— 18 Jun 2010

$a'_k \leadsto a'_l$

▸**Page 118** the end of answer 54(c) ——— 05 Nov 2005

$0 \le \cdots \le c_{k+1} \le \cdots c_k \leadsto 0 \le \min(a_{k+1}, b_{k+1}) \le \min(a_{k+1}, b_{k+1} + b_1 + \cdots + b_k - a_1 - \cdots - a_k) = c_{k+1} \le a_{k+1} \le a_k = c_k + (c_1 + \cdots + c_{k-1}) - (a_1 + \cdots + a_{k-1}) \le c_k$.

▸**Page 118** in part (f) of answer 54 ——— 18 Jun 2010

False ... (17 16 5 5 4). $\leadsto$ False for $\alpha \vee \beta$; for example, $6321 \vee 543 = 633$ in Fig. 52.

Page 118 and 119, simplification of answer 55(a) ——— 11 Jun 2010

Find the smallest k ... $[k = l' + 1]$. $\leadsto$ Find the smallest k with $a_k > b_k$, the smallest l with $a_k > a_{l+1}$, and the smallest m with $a_k - 1 > a_{m+1}$. (Note that $b_k > 0$.) If $a_m > a_{m+1} + 1$, define $\gamma = c_1c_2\ldots$ by $c_k = a_k - [k = m] + [k = m+1]$. Otherwise let $c_k = a_k - [k = l] + [k = m+1]$.

▸**Page 120** line −3 of answer 57 ——— 22 Sep 2008

40–70 $\leadsto$ 49–70

▸**Page 121** line −1 of answer 59 ——— 22 Sep 2008

(see exercise 16) $\leadsto$ (see exercise 15)

Page 122 line −2 of answer 64 ——— 23 Jul 2009

to appear. $\leadsto$ `http://arxiv.org/abs/0907.3873`.

Page 122 line −7 of answer 66 ——— 22 Sep 2008

for all such $\leadsto$ for the number of all such

▸**Page 123** line 5 of answer 68 ——— 22 Sep 2008

No part $\leadsto$ Assume that $n > 1$. Then no part

Page 124 lines 12–13 ——— 11 Sep 2009

The origin of the problem is unknown; see $\leadsto$ The problem reportedly stems from Russia via Bulgaria and Sweden. See also

Page 124 new answer ——— 11 Sep 2009

72. Equivalently, $a_1 < m-1$ [B. Hopkins and J. A. Sellers, *Integers* **7**, 2 (2007), #A19]. [The former answer 72 now becomes number 73.]

Page 124 line 1 of answer 73 (formerly 72) ——— 21 Aug 2007

Swap $\leadsto$ [R. Stanley, 1972 (unpublished).] Swap

Page 125 line 4 of answer 7 ——— 25 Apr 2007

"Partially ordered ... to appear $\leadsto$ *Discrete Math.* **298** (2005), 212–229

Page 131 lines 5–7 ——— 25 Apr 2007

in the paper "Crossings . . . partitions" $\rightsquigarrow$ in a paper
(preprint, 2005) $\rightsquigarrow$ [*Transactions of the Amer. Math. Soc.* **359** (2007), 1555–1575]

Page 131 line −2 of answer 27 ——— 05 Nov 2005

S. Fomin, $\rightsquigarrow$ A. Berele, *J. Combinatorial Theory* **A43** (1986), 320–328; S. Fomin,

▸**Page 132** line 3 of answer 29 ——— 06 Jun 2008

example of Fig. 35 $\rightsquigarrow$ example of Fig. 56

▸**Page 132** line 23 of answer 29 ——— 27 Jun 2006

[The formula $\rightsquigarrow$ *Notes:* The formula

▸**Page 134** last line of answer 35 ——— 19 Oct 2008

58 (1954) $\rightsquigarrow$ **58** (1952)

▸**Page 138** line −4 of answer 53 ——— 06 Jun 2008

See P. L. Chebyshev $\rightsquigarrow$ [See P. L. Chebyshev

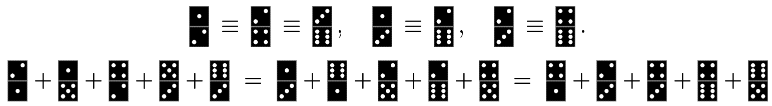
Page 141 lines 2 and 3 of answer 68 ——— 18 Jan 2007

The maximum level is also equal to n. $\rightsquigarrow$ Then the values of $l + 1 = n$ and $b = mn_1 + (m-1)n_2 + \cdots + n_m$ reach their maximum. [Thus it's best to choose names of the multiset elements so that $n_1 \le n_2 \le \cdots \le n_m$.]

▸**Page 142** bottom line of answer 76 ——— 17 Jun 2010

exercise 75 $\rightsquigarrow$ exercise 73

▸**Page 143** replacement for domino displays in answer 82 ——— 17 Sep 2005



Page 144 and following ——— 23 Jun 2005

Miscellaneous changes to the existing index of Volume 4 Fascicle 3 are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.