☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ ERRATA TO VOLUME 4 FASCICLE 0 ☆☆☆☆

This document is a transcript of the notes that I have been making in my personal copy of *The Art of Computer Programming*, Volume 4, Fascicle 0, since it was first printed in 2008.

Four levels of updates — "errors," "amendments," "plans," and "improvements" — appear, indicated by four different typographic conventions:

▸**Page 666** line 1 ________________ 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \rightsquigarrow n+1$

Page 666 line 2 ________________ 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . .' leading to the date on which I recorded the plan in my files.

Page 666 line 4 ________________ 10 Jan 1938

The fourth and final category — indicated by page and line number in smaller, slanted type — consists of minor corrections or improvements that most readers don't want to know about, because they are so trivial. You wouldn't even be seeing these items if you hadn't specifically chosen to print the complete errata list in all its gory details. Are you sure you wanted to do that?

The ultimate, glorious, future editions of Volumes 1–5 are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 4B, Stanford University, Stanford CA 94305-9045, or by email to `taocp@cs.stanford.edu`*. (Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although I'm working full time on*

Volume 4 these days, I will try to reply to all such messages within a year of receipt. Current news about The Art of Computer Programming is posted on

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

and updated regularly.

— Don Knuth, February 2008

He did a lot of editing. That's how he liked to work.
Sometimes he even made alterations between printings.
— P. D. JAMES, *The Lighthouse* (2005)

INTRO TO COMBINATORIAL ALGS, BOOLEAN FUNCS

Last updated 1 December 2010

Most of these corrections have already been made in recent printings.

Important note: The changes below include nearly every difference between the paperback fascicle and the first printing of Volume 4A, published in January 2011. All subsequent changes can be found in the errata list for that volume.

▸**Page ∞** back cover line 10 ——— 07 Apr 2009

problems.All ⇝ problems. All

Page iii line 15 ——— 22 Dec 2008

on combinatorial algorithms ⇝ about combinatorial searching

▸**Page iii** line −10 ——— 27 Feb 2009

underly ⇝ underlie

Page iv line 15 ——— 28 Oct 2008

I shall happily pay a finder's fee of \$2.56 ⇝ I happily offer a "finder's fee" of \$2.56

Page iv lines 19 and 20 ——— 28 Oct 2008

reward you with immortal glory instead of mere money ⇝ do my best to give you immortal glory

Page ix line 9 ——— 28 Oct 2008

cheerfully pay ⇝ cheerfully award

▸**Page ix** line 17 ——— 31 Oct 2008

(1995) ⇝ (1994)

▸**Page x** line 4 ——— 31 Oct 2008

(1968) ⇝ (1969)

Page 3 line −13 ——— 13 Feb 2008

[The lack of accents in *Recreations* ... is *not* an error: No accents were put onto the title page of Ozanam's book, or on its *planches* of illustrations, until the 1770 edition.]

Page 8 line 20 ——— 09 Oct 2010

Annals of Math. ⇝ *Annals of Math.* (2)

Page 8 line 22 ——— 19 Jul 2009

coding and cryptography ⇝ discrete geometry and error-correcting codes

Page 13 replacement for lines −8 and −7 ——— 03 Dec 2009

- C_n has n edges v — $((v \bmod n)+1)$ for $1 \le v \le n$, when $n \ge 1$; it is a graph only when $n \ge 3$ (but C_1 and C_2 are multigraphs).

▸**Page 24** lines 5 and 6 ——— 24 Sep 2009

the smallest circle that passes through u and v does not enclose $\leadsto$ there's a circle passing through u and v that does not enclose

Page 28 in (51) ——— 25 Mar 2009

$K_2 \,\square\, K_2 \,\square \cdots \square\, K_2 \leadsto P_2 \,\square\, P_2 \,\square \cdots \square\, P_2$

▸**Page 30** use a more robust version of Algorithm H ——— 04 Jan 2010

H1. [Set the c's.] Start with $k \leftarrow d_1$ and $j \leftarrow 0$. Then while $k > 0$ do the following operations: Set $j \leftarrow j+1$; while $k > d_{j+1}$, set $c_k \leftarrow j$ and $k \leftarrow k-1$. Terminate successfully if $j = 0$ (all d's are zero).

H2. [Find n.] Set $n \leftarrow c_1$. Terminate successfully if $n = 0$; terminate unsuccessfully if $d_1 \geq n > 0$.

H3. [Begin loop on j.] Set $i \leftarrow 1$, $t \leftarrow d_1$, $r \leftarrow c_t$, and $j \leftarrow d_n$.

H4. [Generate a new edge.] Set $c_j \leftarrow c_j - 1$ and $m \leftarrow c_t$. Create the edge $n \longrightarrow m$, and set $d_m \leftarrow d_m - 1$, $c_t \leftarrow m - 1$, $j \leftarrow j - 1$. If $j = 0$, return to step H2. Otherwise, if $m = i$, set $i \leftarrow r + 1$, $t \leftarrow d_i$, and $r \leftarrow c_t$ (see exercise 104); repeat step H4. ▮

▸**Page 32** line 8 ——— 06 Apr 2010

1993 $\leadsto$ 1994

Page 33 new (60) to match (57) better ——— 09 Dec 2009

$$(x_1 \oplus x_2 \oplus x_3) \wedge (x_2 \oplus x_4 \oplus x_6) \wedge (x_3 \oplus x_4 \oplus x_7) \wedge (x_3 \oplus x_5 \oplus x_6) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_4) \qquad (60)$$

Page 36 the rating of exercise 2 ——— 30 Mar 2009

[*18*] $\leadsto$ [*20*]

Page 37 the rating of exercise 18 ——— 09 Jul 2009

[*M23*] $\leadsto$ [*M26*]

Page 37 the rating of exercise 23 ——— 09 Jul 2009

[*M21*] $\leadsto$ [*M23*]

▸**Page 39** in exercise 43 ——— 31 Jul 2008

[A printing error has put a spurious blob of blank ink on the rightmost illustration in some copies (perhaps not in all).]

Page 41 line 2 of exercise 72 ——— 29 Sep 2010

the *ancestors* $\leadsto$ the (inclusive) *ancestors*

▸**Page 41** line 5 of exercise 75 ——— 09 Jun 2008

$t = j \leadsto t \leftarrow j$

Page 42 new rating for exercise 78 ——— 28 Apr 2009

M26 $\leadsto$ *M27*

Page 43 new rating for exercise 105 ——— 07 Dec 2009

M34 $\leadsto$ *M38*

Page 44 in exercise 119 because of the new (60) ——— 09 Dec 2009

$(\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \leadsto (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_4)$

Page 49 in the heading line of Table 1 ——— 12 Jun 2008

Notation(s) $\leadsto$ New and old notation(s)

Page 49 in Table 1 ——— 08 Jan 2009

line '0011': Left projection $\leadsto$ Left projection; first dictator
line '0101': Right projection $\leadsto$ Right projection; second dictator

Page 52 lines 16–17 ——— 17 Dec 2008

multilinear representation $\leadsto$ *multilinear representation* or *exclusive normal form*

Page 55 line 7 ——— 07 Oct 2008

a best disjunctive normal form $\leadsto$ a shortest disjunctive normal form

Page 55 line 8 ——— 03 Aug 2009

NP-complete $\leadsto$ NP-hard

▸**Page 56** line 11 ——— 21 Apr 2008

2^n truth tables $\leadsto$ 2^n-bit truth tables

Page 60 line 22 and 23 ——— 19 Jul 2008

in a paper ... 284. $\leadsto$ in papers by W. F. Dowling and J. H. Gallier, *J. Logic Programming* **1** (1984), 267–284; M. G. Scutellá, *J. Logic Programming* **8** (1990), 265–273.

Page 61 in (42) ——— 07 Oct 2008

$u \leadsto u.$ (where 'u' appears the second time)

▸**Page 62** line −10 ——— 01 Feb 2010

(1993) $\leadsto$ (1994)

Page 62 (not an error) ——— 03 Jun 2008

[The index refers to a "pun resisted" on this page. I was thinking of "co-medians"...]

Page 63 lines 16 and 17 ——— 18 Jun 2008

if that was our preference $\leadsto$ if we wanted to

▸**Page 75** line 21 ——— 02 Mar 2009

McCullough $\leadsto$ McCulloch

Page 90 line 1 of exercise 79 ——— 13 Dec 2009

A *subgraph* $\leadsto$ An induced *subgraph*

Page 90 line 6 of exercise 80 ——— 13 Dec 2009

Find a subgraph $\leadsto$ Find an induced subgraph

Page 91 the rating of exercise 83 ——— 18 Jan 2010

[*28*] ⇝ [*38*]

Page 92 line -2 ——— 02 May 2008

for all $k \geq 0$ ⇝ for $0 \leq k \leq n$

Page 102 line -17 ——— 14 Feb 2009

from t: ⇝ from t (see 7.1.3–(83)):

▸**Page 104** replacement for lines 18–34 ——— 09 Aug 2010

tubes: They had four kinds of gates, $\mathrm{NOT}(f)$, $\mathrm{NAND}(f,g)$, $\mathrm{OR}(f_1,\ldots,f_k)$, and $\mathrm{AND}(f_1,\ldots,f_k)$, respectively costing 1, 2, k, and 0. Every input to NOT, NAND, or OR could be either a variable, or the complement of a variable, or the result of a previous gate; every input to AND had to be the output of either NOT or NAND that wasn't also used elsewhere.

With those cost criteria, a function might not have the same cost as its complement. One could, for instance, evaluate $x \wedge y$ as $\mathrm{AND}\bigl(\mathrm{NOT}(\bar{x}),\mathrm{NOT}(\bar{y})\bigr)$, with cost 2; but the cost of $\bar{x} \vee (\bar{y} \wedge \bar{z}) = \mathrm{NAND}(x,\mathrm{OR}(y,z))$ was 4 while its complement $x \wedge (y \vee z) = \mathrm{AND}\bigl(\mathrm{NOT}(\bar{x}),\mathrm{NAND}(\bar{y},\bar{z})\bigr)$ cost only 3. Therefore the Harvard researchers needed to consider 402 essentially different classes of 4-variable functions instead of 222 (see the answer to exercise 7.1.1–125). Of course in those days they worked mostly by hand. They found $V(f) < 20$ in all cases, except for the 64 functions equivalent to $S_{0,1}(w,x,y,z) \vee \bigl(S_2(w,x,y) \wedge z\bigr)$, which they evaluated with 20 control grids as follows:

$$
\begin{aligned}
&g_1 = \mathrm{AND}(\mathrm{NOT}(\bar{w}),\mathrm{NOT}(\bar{x})),\ \ g_2 = \mathrm{NAND}(\bar{y},z),\\
&\qquad g_3 = \mathrm{AND}(\mathrm{NOT}(w),\mathrm{NOT}(x));\\
&f = \mathrm{AND}\bigl(\mathrm{NAND}(g_1,g_2),\mathrm{NAND}(g_3,\mathrm{AND}(\mathrm{NOT}(\bar{y}),\mathrm{NOT}(\bar{z}))),\\
&\qquad\qquad \mathrm{NOT}(\mathrm{AND}(\mathrm{NOT}(g_3),\mathrm{NOT}(\bar{y}),\mathrm{NOT}(z))),\\
&\qquad\qquad\qquad \mathrm{NOT}(\mathrm{AND}(\mathrm{NOT}(g_1),\mathrm{NOT}(g_2),\mathrm{NOT}(g_3)))\bigr).
\end{aligned}
\tag{20}
$$

Page 113 replacement for (43) and the preceding line ——— 05 Jul 2010

aren't always so lucky, but a chain of 22 steps does emerge; and David Stevenson has shown that only 21 steps are actually needed if we choose x_{10} non-greedily:

$$
\begin{array}{lll}
x_5 = x_2 \oplus x_3, & x_{12} = x_1 \wedge x_2, & \bar{a} = x_{19} = x_{15} \oplus x_{18},\\
x_6 = \bar{x}_1 \wedge x_4, & x_{13} = x_9 \wedge \bar{x}_{12}, & \bar{b} = x_{20} = x_{11} \wedge \bar{x}_{13},\\
x_7 = x_3 \wedge \bar{x}_6, & x_{14} = \bar{x}_3 \wedge x_{13}, & \bar{c} = x_{21} = \bar{x}_8 \wedge x_{11},\\
x_8 = x_1 \oplus x_2, & x_{15} = x_5 \oplus x_{14}, & \bar{d} = x_{22} = x_9 \wedge \bar{x}_{16},\\
x_9 = x_4 \oplus x_5, & x_{16} = x_1 \oplus x_7, & \bar{e} = x_{23} = x_6 \vee x_{14},\\
x_{10} = x_3 \vee x_9, & x_{17} = x_1 \vee x_5, & \bar{f} = x_{24} = \bar{x}_8 \wedge x_{15},\\
x_{11} = x_6 \oplus x_{10}, & x_{18} = x_6 \oplus x_{13}, & g = x_{25} = x_7 \vee x_{17}.
\end{array}
\tag{43}
$$

▸**Page 127** in exercise 37 ——— 15 May 2009

line 2 of part (b): Prove that ⇝ Prove for fixed m that

line 1 of part (c): Given n, ⇝ Given $n > 1$,

▸**Page 128** line 3 of exercise 42 ——— 06 Apr 2010

$(u_1 \wedge v_0) \rightsquigarrow (v_1 \wedge u_0)$

Page 128 in exercise 43 ——— 08 Mar 2010

line 2 of part (a): k is odd $\rightsquigarrow$ $k \geq 1$ is odd
line 1 of part (b): transducer $\rightsquigarrow$ transducer with $|Q| = 2$

Page 130 line 4 of exercise 67 ——— 07 Mar 2010

among X's best. $\rightsquigarrow$ legal and minimizes X's worst-case outcome.

▸**Page 131** line 1 of exercise 76(c) ——— 08 May 2010

$[j > l] \rightsquigarrow [j < l]$

Page 135 line 3 of answer 7 ——— 19 Jul 2009

$(0, 1, 2, 3, 4, 5, 6, 5, 6, \ldots, 5, 6, 5, 4, 3, 2, 1, 0)$. This sequence $\rightsquigarrow$
$(d_0, d_1, \ldots, d_{2n}) = (0, 1, 2, 3, 4, 5, 6, 5, 6, \ldots, 5, 6, 5, 4, 3, 2, 1, 0)$, which

▸**Page 141** end of answer 43 ——— 30 Dec 2009

girth 4 $\rightsquigarrow$ girth 3

Page 142 first line of answer 60 ——— 26 Apr 2009

Set $A =$ $\rightsquigarrow$ Let $A =$

▸**Page 144** line 7 of answer 78 ——— 26 Apr 2009

all multiples of 4. $\rightsquigarrow$ all even.

Page 147 line 2 of answer 104 ——— 03 Dec 2009

Variables r and i mark the starting and ending row $\rightsquigarrow$ Variables i and r delimit the available rows

▸**Page 147** replacement for all but the first paragraph of answer 105 ——— 04 Jan 2010

Let $a_k = d_1 + \cdots + d_k - c_1 - \cdots - c_k + k$, and suppose that we reach $a_k > 0$ in step H2 for some $k \leq s$. Let A_j, C_j, D_j, N, and S be the numbers that correspond to a_j, c_j, d_j, n, and s *before* steps H3 and H4; thus $N = n + 1$, $D_j = d_j + (0 \text{ or } 1)$, etc. We want to prove that $A_K > 0$ for some $K \leq S$.

Steps H3 and H4 have removed row N and the bottommost remaining q cells in column t, for some $t \geq S$ and $q > 0$, together with the rightmost cells in rows 1 through p. If $p > 0$ we have $C_{t+1} = p$. Let $r = D_N = p + q$, $u = C_t$, and $v = c_t$. Notice that $D_j = t$ for $p < j \leq u$, and $C_j = N$ for $1 \leq j \leq r$; also $A_j = a_j$ for $1 \leq j \leq p$.

If k is minimal we have $1 \leq a_k \leq d_k - c_k + 1$, hence $c_k \leq d_k$. If $D_k > t$ then $k \leq p$ and $A_k = a_k$. If $D_k < t$ it follows that $A_k = a_k + r - \min(k, r) \geq a_k$, because $k \leq D_k$. Thus we may assume that $D_k = t$.

Suppose $t > S$; hence $u \leq S$. For $k < j \leq u$ we have $d_j \geq D_j - 1 = t - 1 \geq d_k - 1 \geq c_k - 1 \geq c_j - 1$. Thus $a_u \geq a_k > 0$. But $A_u = a_u$, because $r \leq u \leq S < t$. We may therefore assume that $t = S$. Suppose $k < t$; then $c_k = d_k = t$, because $S \leq c_k \leq d_k \leq t$. But $r = t$ leads to $c_k = N - 1$ and a contradiction; and $r < t$ leads to $u = t$, from which it follows that $A_t > A_{t-1} = a_{t-1} - 1 \geq 0$.

(Deep breath.) OK; we've reduced the problem to cases with $k = t = S$. Hence $t = s \leq c_t \leq d_t \leq D_t = t$, and we have $a_t = a_{t-1} + 1$. Consequently $a_{t-1} = 0$.

In fact we can show by induction on $t - j$ that $a_j = 0$ for $p \leq j < t$: If $a_{j+1} = 0$ then $0 \leq a_j = c_{j+1} - t - 1 \geq q - 1 \geq 0$, because $c_{j+1} \geq t + q$ when $p \leq j < t - 1$.

If $p < t-1$, this argument proves that $q = 1$ and $c_r = N - 1 = t+1$. We conclude that, regardless of p, we must have $q = 1$, $N = t+2$, $D_j = t+1$ for $1 \le j \le p$, $D_j = t$ for $p < j \le t+1$, and $D_N = p+1$. Algorithm H does actually change this "good" sequence into a "bad" one; but $D_1 + \cdots + D_N = 2p + t(t+1) + 1$ is odd.

[Note: Now answers 106–145 move about 10 lines down; many index entries change.]

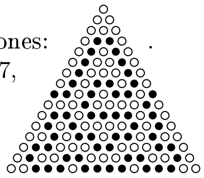
▸**Page 148** replacement for first paragraph of answer 108 ———— 04 Jan 2010

108. We may assume that $d_1^+ \ge \cdots \ge d_n^+$; the in-degrees d_k^- need not be in any particular order. Apply Algorithm H to the sequence $d_1 \ldots d_n = d_1^+ \ldots d_n^+$, but with the following changes: Step H2 becomes "[Done?] Terminate successfully if $d_1 = n = 0$; terminate unsuccessfully if $d_1 > n$." In step H3, change "$j \leftarrow d_n$" to "$j \leftarrow d_n^-$," and terminate unsuccessfully if $j > c_1$. In step H4, omit "$c_j \leftarrow c_j - 1$"; change "$n \longrightarrow m$" to "$m \longrightarrow n$"; and set $n \leftarrow n-1$ just before returning to H2. An argument like Lemma M and Corollary H justifies this approach.

Page 150 line 2 of answer 133 ———— 28 Apr 2009

[Can it be drawn with fewer than 12 crossings?] $\leadsto$ [In fact, Markus Chimani used branch-and-cut methods in 2009 to prove that it cannot be drawn with fewer than 12 crossings.]

Page 166 replacement for answer 68 ———— 25 Aug 2008

68. Here is one of the 258,594 solutions for $n = 15$ that has 59 black stones:  . (The answers for $1 \le n \le 15$ are respectively 2, 3, 4, 6, 8, 11, 14, 18, 23, 27, 33, 39, 45, 52, 59. The prime implicants for these functions can be represented by fairly small ZDDs; see Section 7.1.4.)

Page 168 line 1 of answer 81 ———— 13 Dec 2009

a subgraph $\leadsto$ an induced subgraph

▸**Page 169** replacement for answer 83 ———— 18 Jan 2010

83. Consider first the Boolean (two-vertex) case, and let an optimum solution be obtainable by the recursive rules $u_0 \leftarrow v_0$ and $u_j \leftarrow f_{t+2-j}(u_{j-1}, v_j, \ldots, v_t)$ for $1 \le j \le t$, where each f_k is a suitable Boolean function of k variables. The first function $f_{t+1}(v_0, v_1, \ldots, v_t)$ actually depends on its "most remote" variable v_t, because we must have $f_{2k+1}(0,1,1,0,1,0,1,\ldots,0,1,0,x) = x$ when $\rho = 1 - \epsilon$ and $k \ge 2$.

One suitable function f_{t+1} can be obtained as follows: Let $f_{t+1}(0, v_1, \ldots, v_t) = 0$ if $v_1 = 0$. Otherwise let the "runs" of the input sequence be

$$v_0 v_1 \ldots v_t = 01^{a_k}0^{a_{k-1}} \ldots 1^{a_2}0^{a_1} \qquad \text{or} \qquad 01^{a_k}0^{a_{k-1}} \ldots 1^{a_3}0^{a_2}1^{a_1},$$

where $a_k, \ldots, a_1 \ge 1$, and let $\alpha_j = 2 \dot{-} a_j\rho = \max(0, 2 - a_j\rho)$ for $1 \le j \le k$. Then

$$f_{t+1}(0, v_1, \ldots, v_t) = \bigl[\alpha_k \dot{-} (\alpha_{k-1} \dot{-} (\cdots \dot{-} (\alpha_2 \dot{-} (1 \dot{-} a_1\rho))\cdots)) = 0\bigr].$$

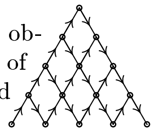
Also let $f_{t+1}(1, v_1, \ldots, v_t) = \bar{f}_{t+1}(0, \bar{v}_1, \ldots, \bar{v}_t)$, so that f_{t+1} is self-dual.

With a somewhat delicate proof one can show that f_{t+1} is also monotone.

Therefore, by Theorem P, we can apply f_{t+1} componentwise to the labels of an arbitrary median graph, always staying within the graph.

Page 173 replacement for answer 109(d) ——— 02 May 2008

(d) True, because max and min satisfy these distributive laws. (In fact, we obtain a distributive *mixed-radix majorization lattice* in a similar way from the set of all n-tuples $a_1 \ldots a_n$ with $0 \le a_j < m_j$ for $1 \le j \le n$. R. P. Stanley has observed that Fig. 8 is also the lattice of order ideals of the triangular grid shown here.)



Page 174 line 2 ——— 14 Feb 2009

Then $\rightsquigarrow$ Thus

Page 174 line 7 of answer 111(c) ——— 14 Feb 2009

where f is the function derived in part (a) $\rightsquigarrow$ derived from these larger probabilities as in part (a)

▸**Page 174** line 11 of answer 111(c) ——— 23 Sep 2008

$F_k(p \ldots, p) \rightsquigarrow F_k(p, \ldots, p)$

Page 177 lines 11–12 of answer 121 ——— 14 Feb 2009

the ten cases when $m = 3$ and $n = 6$ are $\rightsquigarrow$ the following ten cases arise when $m = 3$ and $n = 6$:

Page 179 replacement for answer 132(c) ——— 12 Mar 2010

(c) The argument in part (a) proves that $f(x)$ is bent if and only if $s(y) = 2^{n/2}(-1)^{g(y)}$ for some Boolean function $g(y)$. This function g, the Fourier/Hadamard transform of f, is also bent, because $\sum_y (-1)^{g(y)+w\cdot y} = 2^{-n/2}\sum_{x,y}(-1)^{f(x)+x\cdot y+w\cdot y} = 2^{n/2}\sum_x (-1)^{f(x)}[x = w] = 2^{n/2}(-1)^{f(w)}$ for all w. The hint now tells us that we have $\sum_y (-1)^{g(y)+g(y\oplus z)} = 0$ for all nonzero z, and the same holds for f.

Conversely, assume that $f(x)$ satisfies the stated condition. Then we have

$$s(y)^2 = \sum_{x,t}(-1)^{f(x)+x\cdot y+f(x\oplus t)+(x\oplus t)\cdot y} = \sum_t (-1)^{t\cdot y}\sum_x (-1)^{f(x)+f(x\oplus t)} = 2^n$$

for all y.

Page 180 in the final paragraph of answer 132 ——— 12 Mar 2010

line 6: $2^{n-1} - 2^{(n-3)/2}$ from all affine functions when g and $h \rightsquigarrow 2^{n-1} - 2^{(n-1)/2}$ from all affine functions when g and $g \oplus h$

line 9: 108.] $\rightsquigarrow$ 108. S. Kavut and M. Diker Yücel, *Information and Computation* **208** (2010), 341–350, have achieved distance $2^8 - 14$ when $n = 9$.]

Page 181 final line of answer 4 ——— 25 Nov 2010

527.] $\rightsquigarrow$ B. Doerr improved α to $-1/\lg(\sqrt{3} - 1)$ in Math. Sem. Univ. Kiel Tech. Report 99-28 (1999).]

Page 182 line 1 ——— 05 Mar 2009

to an appropriate bit vector of weight 1 $\rightsquigarrow$ to its unique footprint vector (which contains exactly one 1)

▸**Page 182** replacement for step U6 ——— 04 Mar 2009

U6. [Update $U(f)$ and $\phi(f)$.] If $U(f) = \infty$, set $c \leftarrow c - 1$, $\phi(f) \leftarrow v$, $U(f) \leftarrow u$, and put f into list u. Otherwise if $U(f) > u$, move f from list $U(f)$ to list u and set $\phi(f) \leftarrow v$, $U(f) \leftarrow u$. Otherwise if $U(f) = u$, set $\phi(f) \leftarrow \phi(f) \mid v$. ▮

▶**Page 182** answer 12 — 02 May 2010

$x_5 = x_3 \wedge x_2$ ⇝ $x_5 = x_3 \wedge x_4$

Page 184 addendum to answer 21 — 09 Aug 2010

The minimum cost of their hardest function (20) is still unknown, but a vacuum-tube variant of the footprint method shows that $V(f) \le 18$:

$$\mathrm{OR}\big(\mathrm{AND}(\mathrm{NAND}(\bar{x}_1, x_2), \mathrm{NAND}(x_1, \bar{x}_2), \mathrm{NAND}(x_1, \bar{x}_4), \mathrm{NAND}(x_3, x_4)\big), \\ \mathrm{AND}\big(\mathrm{NAND}(x_1, x_2), \mathrm{NAND}(\bar{x}_1, \bar{x}_2), \mathrm{NAND}(x_3, \bar{x}_4), \mathrm{NAND}(\bar{x}_3, x_4)\big)\big).$$

Although they failed to find this particular construction, the Harvard researchers did remarkably well, in some cases beating the footprint heuristic by as many as 6 grids!

Page 185 answer 36 — 15 May 2009

line 2: optimum depth and Q_n has depth $\le \lceil \lg n \rceil + 1$: ⇝
$D(y_j) \le \lceil \lg n \rceil$ for $1 \le j \le n$ and Q_n has $D(y_j) \le \lceil \lg n \rceil + [j \ne n]$:
lines 7–9: To prove validity ... power of 2. ⇝

Page 186 lines −6 and −5 of answer 36 — 15 May 2009

use the otherwise-forbidden $P'_{\lfloor n/2 \rfloor}$ construction for Q_n when $n = 2^m + 1$, ⇝
replace Q_{2n+1} by (Q_{2n} and $y_{2n+1} = y_{2n} \wedge x_{2n+1}$) when n is a power of 2,

Page 186 in answer 37(c) — 15 May 2009

line 3: $d(n) > d(n-1) > 0$ ⇝ $d(n) > d(n-1)$ and $n > 2$
line 4: minimum occurs ⇝ minimum with minimal cost occurs

▶**Page 186** last line of answer 37(c) — 15 May 2009

$n - \lfloor \frac{2}{3}(n - \lfloor \lg n \rfloor + 3) \rfloor + 2$ ⇝ $n - \lfloor \lg \frac{2}{3}(n - \lfloor \lg n \rfloor + 3) \rfloor + 2$

▶**Page 189** line 3 of answer 43 — 28 Apr 2010

$d_{j+1}(q) = d(d_j(q), a_j)$ ⇝ $d_j(q) = d(d_{j-1}(q), a_j)$
$d_{j+1} = d_j \circ d^{(a_j)}$ ⇝ $d_j = d_{j-1} \circ d^{(a_j)}$

Page 189 line 13 of answer 43 — 08 Mar 2010

$b_j = \bar{q}_j \wedge a_j = \bar{d}_{(j-1)0} \wedge a_j$ ⇝ $b_1 = a_1$, and $b_j = \bar{q}_j \wedge a_j = \bar{d}_{(j-1)0} \wedge a_j$ for $j > 1$

Page 189 last line of answer 43 — 04 Apr 2010

not $6p_4 + 4 = 28$ ... 5.) ⇝ $6p_{n-1} + (n-1) = 28$; the actual depth is 4, not $2\lceil \lg(n-1) \rceil + 1 = 5$. Notice that the straightforward chain $b_j = a_j \wedge \bar{b}_{j-1}$ for $1 < j \le n$ also solves problem (a); it wins on cost, but has depth $n - 1$.)

▶**Page 189** in answer 44 — 04 Apr 2010

lines 12 and 13: The total cost is 20, and the maximum depth is 5. ⇝ The total cost is 20; the maximum depth, 6, occurs in the computation of z_3.
line 18: clearly superior to the conditional-sum method (which has the same depth $2m+1$) ⇝ superior to the conditional-sum method (which however has depth $2m+1$, not $2m+2$)

Page 190 replacement for last line of answer 52 — 05 Jul 2010

(These upper bounds are quite weak when n is small. For example, we know that $C(n) = (0, 1, 4, 7, 12)$ when $n = (1, 2, 3, 4, 5)$; and Eq. 7.1.1–(16) gives $C(n+1) \le 2C(n) + 2$, so that $C(6) \le 26$, $C(7) \le 54$, etc.)

▸**Page 191** replacement for last line of answer 57 ———— 17 Aug 2009

1010 $\mapsto$, 1011 $\mapsto$, 1100 $\mapsto$, 1101 $\mapsto$, 1110 $\mapsto$, 1111 $\mapsto$.

Page 191 replacement for first six lines of answer 54 ———— 02 Oct 2010

54. The greedy-footprint heuristic gives a chain of length 14:

$$\begin{aligned}
&x_5 = x_1 \oplus x_3, && x_{10} = x_4 \wedge \bar{x}_5, && f_3 = x_{15} = \bar{x}_8 \wedge x_9,\\
&x_6 = x_2 \oplus x_3, && x_{11} = x_4 \oplus x_5, && f_4 = x_{16} = x_4 \wedge x_8,\\
&x_7 = x_1 \wedge x_2, && x_{12} = x_6 \wedge x_{11}, && f_5 = x_{17} = x_7 \wedge x_9,\\
&x_8 = x_1 \wedge \bar{x}_6, && f_1 = x_{13} = \bar{x}_7 \wedge x_{12}, && f_6 = x_{19} = x_6 \wedge x_{10}.\\
&x_9 = x_4 \wedge x_5, && f_2 = x_{14} = \bar{x}_6 \wedge x_{10},
\end{aligned}$$

Page 191 in answer 55 ———— 05 Jul 2010

line 2: 30 $\leadsto$ 29
line 7: $x_7 \wedge x_{12} \leadsto x_8 \wedge x_{12}$

Page 191 additional material for answer 57 ———— 05 Jul 2010

[David Stevenson, believing that it would be better to have a solution in which nondigits never masquerade as digits, has come up with another chain of length 13, namely

$$\begin{aligned}
&x_5 = \bar{x}_1 \wedge x_4, && \bar{d} = x_{10} = \bar{x}_6 \wedge x_8, && \bar{c} = x_{15} = \bar{x}_2 \wedge x_6,\\
&x_6 = x_3 \wedge \bar{x}_5, && \bar{a} = x_{11} = \bar{x}_3 \wedge x_7, && x_{16} = c_9 \oplus x_{10},\\
&x_7 = x_2 \oplus x_5, && \bar{b} = x_{12} = x_2 \wedge \bar{x}_8, && g = x_{17} = x_1 \oplus x_{16},\\
&x_8 = x_3 \oplus x_7, && \bar{f} = x_{13} = \bar{x}_2 \wedge x_9,\\
&x_9 = x_3 \vee x_5, && \bar{e} = x_{14} = x_4 \vee x_{11},
\end{aligned}$$

for which a, ..., g have the truth tables `b7f3`, `f9fc`, `dfcf`, `b6f3`, `a2a2`, `8fcf`, `3ec0`, and

1010 $\mapsto$, 1011 $\mapsto$, 1100 $\mapsto$, 1101 $\mapsto$, 1110 $\mapsto$, 1111 $\mapsto$.

Is there a 13-step chain for (44) that produces 16 *distinct* outputs?]

[*Note:* The foregoing change does not fit in the paperback fascicle, but it will be made in the hardcover Volume 4A.]

Page 192 replacement for lines 2–6 of answer 59 ———— 02 Oct 2010

$$\begin{aligned}
&x_5 = x_2 \oplus x_3, && x_{11} = x_2 \vee x_7, && x_{17} = \bar{x}_6 \wedge x_8,\\
&x_6 = x_1 \oplus x_4, && x_{12} = x_2 \wedge \bar{x}_6, && f_1 = x_{18} = x_{11} \oplus x_{17},\\
&x_7 = x_1 \oplus x_3, && x_{13} = x_3 \wedge x_4, && f_2 = x_{19} = x_{10} \wedge \bar{x}_{14},\\
&x_8 = x_4 \vee x_5, && x_{14} = x_4 \wedge x_5, && f_3 = x_{20} = x_9 \oplus x_{16},\\
&x_9 = x_6 \wedge x_8, && x_{15} = x_5 \wedge x_{10}, && f_4 = x_{21} = x_{12} \oplus x_{15}.\\
&x_{10} = x_7 \vee x_9, && x_{16} = x_2 \wedge \bar{x}_{13},
\end{aligned}$$

Page 192 new solution to exercise 61 ———— 07 Jun 2010

61. The following 17-gate solution by David Stevenson generalizes to $8m+1$ gates for addition mod 2^m+1: $u_0 = x_0 \wedge y_0$, $v_0 = x_0 \oplus y_0$, $u_1 = x_1 \wedge y_1$, $v_1 = x_1 \oplus y_1$, $t_1 = v_1 \wedge u_0$, $t_2 = v_1 \oplus u_0$, $c_2 = u_1 \vee t_1$; $u_2 = x_2 \wedge y_2$, $t_3 = x_2 \vee y_2$, $t_4 = t_3 \vee c_2$; $t_5 = t_2 \vee v_0$, $t_6 = t_5 \wedge t_4$, $t_7 = t_6 \vee u_2$; $t_8 = t_7 \wedge \bar{v}_0$, $z_0 = t_7 \oplus v_0$, $z_1 = t_2 \oplus t_8$; $z_2 = t_4 \oplus t_7$. (Notice that $(x_2x_1x_0)_2 + (y_2y_1y_0)_2 = (u_2t_4t_2v_0)_2 - 4[x=y=4]$. Gilbert Lee has found another 17-step solution if the inputs are represented by 000, 001, 011, 101, and 111.)

Page 193 line 5 of answer 66 ———— 06 Mar 2010

$\leadsto$ or

Page 194 replacement for the first paragraph of answer 67 ———— 07 Mar 2010

67. The best solution known so far, due to David Stevenson in 2010, uses a total of 818 gates (472 AND, 327 OR, 13 NOR, 6 ANDN); see

`http://www-cs-faculty.stanford.edu/~knuth/818-gate-solution`

for the details. After taking care of moves such as w_j and b_j, and cleverly optimizing don't cares, Stevenson essentially ORs together about 200 special positions (such as ) that make $c = 1$, about 200 others (such as ) that make $s = 1$, and about 50 (such as ) that make $m = 1$; then he saves gates by finding common subexpressions among the ANDs that define special positions, and by using the distributive law, etc.

▶**Page 194** line 2 of answer 68 ———— 05 Jul 2010

cost is 2^n times a polynomial in n $\rightsquigarrow$ cost is $\Omega(n2^n)$

Page 195 line 5 of answer 76 ———— 08 May 2010

$(2^m + O(m))$ $\rightsquigarrow$ $(2^m - O(m))$

Page 197 answer 80 ———— 20 Mar 2009

[insert '(a)' at the beginning of the answer; move the journal citation to the end]

▶**Page 199** line -3 ———— 22 Mar 2009

fewer then $\rightsquigarrow$ fewer than

Page 201 and following ———— 05 Feb 2008

Miscellaneous changes to the existing index of Volume 4 Fascicle 0 are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.