☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ EARLIEST ERRATA TO VOLUME 4A ☆☆☆☆

This document is a transcript of the notes that I had been making in my personal copy of *The Art of Computer Programming*, Volume 4A, since it was first printed in 2011.

Four levels of updates — "errors," "amendments," "plans," and "improvements" — appear, indicated by four different typographic conventions:

▸**Page 666** line 1 ———————————————— 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \rightsquigarrow n+1$

Page 666 line 2 ———————————————— 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . .' leading to the date on which I recorded the plan in my files.

Page 666 line 4 ———————————————— 10 Jan 1938

The fourth and final category — indicated by page and line number in smaller, slanted type — consists of minor corrections or improvements that most readers don't want to know about, because they are so trivial. You wouldn't even be seeing these items if you hadn't specifically chosen to print the complete errata list in all its gory details. Are you sure you wanted to do that?

The ultimate, glorious, future editions of The Art of Computer Programming are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 1B, Stanford University, Stanford CA 94305-9015, or by email to `taocp@cs.stanford.edu`. *(Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although*

I'm working full time on Volume 4B these days, I will try to reply to all such messages within a year of receipt. Current news is posted on

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

and updated regularly.

— Don Knuth, February 2022

If there happen to bee any iarre or dissonance, blame not the Printer,
who (I doe assure thee) through his great paines and diligence,
doth heere deliuer to thee a perfect and true Coppie.
If . . . there bee any fault by mee committed,
I desire the skilfull, eyther with courtesie to let the same bee concealed,
or in friendly sort to bee thereof admonished:
and at the next Impression he shall find error reformed:
remembring alwaies, that it is more easie to finde a fault then to amend it.

— WILLIAM BYRD, *Psalmes, Sonets, & songs of sadnes and pietie* (1588)

He did a lot of editing. That's how he liked to work.
Sometimes he even made alterations between printings.

— P. D. JAMES, *The Lighthouse* (2005)

☆☆ COMBINATORIAL ALGORITHMS, PART 1 ☆☆

Last updated 9 February 2023

All of these corrections have been made in subsequent printings.

Page v lines 21 and 22 ——— 18 Jul 2017

number for a 10-letter alphabet … is 5. ⇝ number when we replace 24 by 10 in Chambers's statement is 11,111,111,110; and it's only 3905 when we reduce the parameter to 5.

▸**Page viii** line 9 ——— 11 Nov 2021

Section 7.4.1 ⇝ Section 7.4.1.3

Page viii line 11 ——— 26 May 2011

Section 7.2 ⇝ Section 7.2.2

Page viii line 13 ——— 25 Jul 2020

covering, coloring, and packing ⇝ coloring and packing

▸**Page ix** line 8 ——— 01 Feb 2019

Oystein ⇝ Øystein

Page xii line 16 ——— 16 Jun 2011

a *45* rating ⇝ a *40* rating

Page xii line 20 ——— 30 Jan 2013

creativity. ⇝ creativity. All exercises with ratings of *46* or more are open problems for future research, rated according to the number of different attacks that they've resisted so far.

Page 1 line 18 ——— 23 Apr 2011

(1927) ⇝ (1926)

Page 2 line −17 ——— 01 Jun 2017

3200 memory accesses ⇝ 3500 memory accesses

▸**Page 3** replacement for display (4) ——— 15 May 2020

$$\{\mathtt{abacbdecfgdoersfpgqtuwxvjklonhmirpsjqkhltiunmwvx},\ \mathtt{xvwsquntkigrdapaodgiknqsvxwutmrpoljecfhbcebmfjlh}\} \qquad (4)$$

▸**Page 3** line −12 ——— 30 May 2018

(1725) ⇝ (1723)

▸**Page 3** line −8 (6th printing only) ——— 03 Dec 2017

Fig. ?? ⇝ Fig. 1

Page 6 line −19 ——— 19 Aug 2015

Section 7.2.3, ⇝ Section 7.2.2

▸**Page 9** line 19 ——— 23 Apr 2011

"Can toy problems be useful?" ⇝ "Are toy problems useful?"

▸**Page 10** line 16 ——— 23 Apr 2011

good society ⇝ *good Society*

▸**Page 10** line −12 ——— 31 Jan 2011

(see page ii) ⇝ (see page iv)

Page 11 line 2 after (10) ——— 13 Aug 2015

Section 7.2.2.2 ⇝ Section 7.2.2.1

Page 12 line 18 ——— 03 Aug 2011

in the author's books about TEX and METAFONT ⇝
in *The TEXbook* and *The METAFONTbook*

▸**Page 12** replacement for (13) ——— 11 Jul 2013

`chaos` — `chose` — `chore` — `score` — `store`
— `stare` — `spare` — `space` — `peace`. (13)

Page 13 line 14 ——— 01 Feb 2012

permitted.) ⇝ permitted, and in which repeated edges are allowed too.)

▸**Page 14** lines 6,8,12 (6th printing only) ——— 09 Apr 2017

Figure ?? ⇝ Figure 2

Page 15 lines −19 and −18 ——— 20 Jul 2011

spanning path ⇝ spanning path P_n
spanning cycle ⇝ spanning cycle C_n

▸**Page 15** line −15 ——— 23 Sep 2019

for polyhedra in general ⇝ for convex polyhedra in general

Page 15 line −11 ——— 03 Aug 2015

combinatorial problem of all ⇝ problem of graph theory

Page 15 line −10 ——— 03 Aug 2015

Section 7.3.3 ⇝ Section 7.2.2.4

▸**Page 16** line 26 ——— 23 Apr 2011

the odds ⇝ changing two different letters, the odds

▸**Page 16** line 29 ——— 23 Apr 2011

2056, 1198, 224 ⇝ 2056, 1186, 224

▸**Page 16** line before (22) — 15 May 2011

out be ⇝ out to be

Page 17 lines 5–7 — 04 Mar 2013

Franz Heinen . . . 1836). ⇝ Joseph Gergonne had, however, already posed and solved the problem for *any* k points in the plane [*Annales de mathématiques pures et appliquées* **1** (1811), 292, 375–384 and planche 6].

▸**Page 18** line −8 (6th printing only) — 03 Dec 2017

Figure ?? ⇝ Figure 3

Page 19 line 5 in the right column of Fig. 3 — 23 Apr 2011

Hector MacQueen ⇝ Hector Willard MacQueen

▸**Page 19** line 15 in the right column of Fig. 3 — 23 Apr 2011

Cyrus Bettman Hardman ⇝ Cyrus Bethman Hardman

Page 22 line −20 — 29 Sep 2018

v plus k times ⇝ v's location plus k times

Page 23 line 20 — 11 Nov 2021

Section 7.4.1 ⇝ Section 7.4.1.2

Page 23 line −11 — 13 Sep 2011

cross references ⇝ cross-references

▸**Page 24** line 9 — 23 Apr 2011

$\mathit{plane\_lisa}(360, 250, 15, 0, 360, 0, 250, 0, 0, 2295000)$ ⇝
$\mathit{plane\_lisa}(360, 250, 15, 0, 360, 0, 250, 0, 22950000)$

▸**Page 24** line 14 (6th printing only) — 03 Dec 2017

Fig. ?? ⇝ Fig. 4

▸**Page 25** line 28 — 23 Apr 2011

`3,1,4,2`—`3,0,4,3` ⇝ `3.1.4.2`—`3.0.4.3`

▸**Page 25** line 33 (6th printing only) — 03 Dec 2017

Fig. ?? ⇝ Fig. 5

Page 26 line 11 — 23 Apr 2011

two edges are adjacent in $L(G)$ ⇝ two edges of G are adjacent in $L(G)$

Page 29 line 6 — 30 Dec 2011

graphical; therefore ⇝ graphical, because a (simple) graph cannot contain more than one edge between two vertices. Therefore

▸**Page 31** lines 13 and 14 — 23 Apr 2011

$\mathit{plane\_lisa}(360, 250, 15, 0, 360, 0, 250, 0, 0, 2295000)$ ⇝
$\mathit{plane\_lisa}(360, 250, 15, 0, 360, 0, 250, 0, 22950000)$

▸**Page 32** lines 2 and 3 —— 19 Mar 2018

value of gate `Z45` is an input to gate `R0:7~` $\leadsto$
value of gate `R0:7~` is an input to gate `Z45`

▸**Page 32** line 13 —— 19 Mar 2018

$risc(16) \leadsto risc(8)$

Page 32 line -8 —— 29 Jun 2012

of colors $\leadsto$ of at most k colors

▸**Page 35** lines 15–17 —— 06 Aug 2011

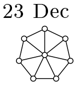
In Section 7.5.1 we'll ... study $\leadsto$ In Sections 7.5.1 and 7.5.5 we'll see that a minimum vertex cover can be discovered quickly in any bipartite graph, or in any hypergraph that is the dual of a graph; we'll also study

Page 36 line 2 of exercise 5 —— 15 Sep 2020

the two k's are exactly k positions apart $\leadsto$ exactly k non-k's appear between the two k's

Page 42 replacement for exercise 88 —— 23 Dec 2015

88. [*M20*] The graph $W_n = K_1 — C_n$ is called the *wheel* with n spokes, when $n \geq 3$. How many cycles does it contain as subgraphs?



W_7

▸**Page 43** line 2 of exercise 107 —— 09 Jan 2019

$d_{2j-1} = d_{2j} \leadsto d_j = d_{j-(-1)^j}$

Page 44 line 2 of exercise 114 —— 23 May 2011

bipartite graph $\leadsto$ bipartite multigraph

Page 45 line 8 of exercise 137 —— 27 Oct 2015

underlying graph $\leadsto$ numbering scheme

Page 46 line 3 of exercise 140 —— 23 Dec 2015

$W_8 \leadsto W_7$

▸**Page 46** replacement for last line of exercise 142 —— 23 May 2011

of order 4 that are considered in exercise 90. *Hint:* Observe that $(n-3)\#(K_3{:}G) = 4\#(K_4{:}G) + 2\#(K_{1,1,2}{:}G) + \#(\overline{K_{1,3}}{:}G) + \#(\overline{K_1 \oplus K_{1,2}}{:}G)$.

Page 48 line 9 before (1) —— 15 Jul 2013

normal English $\leadsto$ normal English for disjoint sets

▸**Page 49** entire page —— 01 Jan 2011

[In the first printing this page was shifted about 1 cm to the left]

Page 49 three entries in Table 1 —— 16 Oct 2014

$x \not\supset y \leadsto x \mathrel{\supset\!\!\!\!\mid} y$
$x \not\subset y \leadsto x \mathrel{\subset\!\!\!\!\mid} y$
$x \not\equiv y \leadsto x \mathrel{\equiv\!\!\!\!\mid} y$

Page 49 in Table 1 for truth table 0010 ——— 06 Feb 2018

but not ⇝ and not; but not

▸**Page 52** line 2 ——— 11 Feb 2012

constants. ⇝ constants, together with the variables $\{x_1,\ldots,x_n\}$.

Page 54 the line before (30) ——— 04 Oct 2018

in (23) are ⇝ in (23) are the terms of

Page 56 lines 4 and 5 after (31) ——— 13 Jul 2011

especially in Section 7.9 ⇝ for instance in Section 7.2.2.2

Page 56 rearrangement of (32) ——— 17 Jun 2013

$$(x_1\vee x_2\vee\bar{x}_3)\wedge(x_2\vee x_3\vee\bar{x}_4)\wedge(x_3\vee x_4\vee x_1)\wedge(x_4\vee\bar{x}_1\vee x_2)$$
$$\wedge(\bar{x}_1\vee\bar{x}_2\vee x_3)\wedge(\bar{x}_2\vee\bar{x}_3\vee x_4)\wedge(\bar{x}_3\vee\bar{x}_4\vee\bar{x}_1)\wedge(\bar{x}_4\vee x_1\vee\bar{x}_2). \qquad (32)$$

[also change 'first seven' to 'last seven' on line −2.]

Page 59 beginning at line 3 after (36) and continuing into page 60 ——— 28 Jul 2016

We use the following ... Algorithm C is similar ⇝
We use the following data structures for clauses c and propositions p:

`CONCLUSION`(c) is the proposition on the right of clause c;

`COUNT`(c) is the number of hypotheses of c not yet asserted;

`TRUTH`(p) is 1 if p is known to be true, otherwise 0;

`LAST`(p) is the last clause in which p is waiting to be asserted;

`PREV`(c) is the previous clause that awaits the same hypothesis as c;

`START`(c) tells where the hypotheses of c appear in `MEM`.

(A "hypothesis" is the appearance of a proposition on the left-hand side of a clause.) An array called `MEM` holds all the left-hand sides of the clauses; if `START`$(c)=l$ and `COUNT`$(c)=k$, the not-yet-asserted hypotheses of clause c are `MEM`$[l+1]$, ..., `MEM`$[l+k]$. We also maintain a stack S_0, S_1, ..., S_{s-1} of all propositions that are known to be true but not yet asserted.

C1. [Initialize.] Set `LAST`$(p)\leftarrow\Lambda$ and `TRUTH`$(p)\leftarrow 0$ for each proposition p. Also set $l\leftarrow s\leftarrow 0$, so that `MEM` and the stack are initially empty. Then for each clause c, having the form (36), set `CONCLUSION`$(c)\leftarrow v$. If $k=0$ and `TRUTH`$(v)=0$, simply set `TRUTH`$(v)\leftarrow 1$, $S_s\leftarrow v$, and $s\leftarrow s+1$. But if $k>0$, set `MEM`$[l+j]\leftarrow u_j$ for $1\le j\le k$, `COUNT`$(c)\leftarrow k$, $l\leftarrow l+k$, `PREV`$(c)\leftarrow$ `LAST`(u_k), and `LAST`$(u_k)\leftarrow c$.

C2. [Prepare to loop.] Terminate the algorithm if $s=0$; the desired core now consists of all propositions whose `TRUTH` has been set to 1. Otherwise set $s\leftarrow s-1$, $p\leftarrow S_s$, and $c\leftarrow$ `LAST`(p). (We'll update the clauses that await p.)

C3. [Done with loop?] If $c=\Lambda$, return to C2. Otherwise set $k\leftarrow$ `COUNT`$(c)-1$, $l\leftarrow$ `START`(c), and $c'\leftarrow$ `PREV`(c).

C4. [Done with c?] If $k = 0$, go to C5. Otherwise set $p \leftarrow$ `MEM`$[l+k]$. If `TRUTH`$(p) = 1$, set $k \leftarrow k-1$ and repeat this step. Otherwise set `COUNT`$(c) \leftarrow k$, `PREV`$(c) \leftarrow$ `LAST`(p), `LAST`$(p) \leftarrow c$, and go to C6.

C5. [Deduce `CONCLUSION`(c).] Set $p \leftarrow$ `CONCLUSION`(c). If `TRUTH`$(p) = 0$, set `TRUTH`$(p) \leftarrow 1$, $S_s \leftarrow p$, $s \leftarrow s+1$.

C6. [Loop on c.] Set $c \leftarrow c'$ and return to C3. ▮

Notice how smoothly the sequential and linked data structures work together, avoiding any need to search for a place to make progress in the calculation. We're doing a bare minimum of work! Algorithm C is similar

▸**Page 60** line −19 — 02 Jan 2012

Scutellá ⇝ Scutellà

Page 62 line −10 — 11 Nov 2021

Section 7.4.1 ⇝ Section 7.4.1.2

▸**Page 68** line 5 — 06 Oct 2018

seven-bit labels ⇝ eight-bit labels

Page 68 line −2 — 18 Jun 2011

Setting $x = y$ ⇝ Setting $y = x$

▸**Page 70** line 1 of step I4 — 11 Jul 2011

`MARK`$(u) \neq s$ ⇝ `MARK`$(u) \neq s$ or `RANK`$(u) \neq 1$

Page 73 line 3 of Theorem F — 07 Oct 2018

for all bit vectors $x_1 \ldots x_t$ ⇝ *for all bit vectors* $x = x_1 \ldots x_t$

▸**Page 74** line 7 — 10 Jun 2011

$u_k \leq u_{k+d}$ ⇝ $u_k \to u_{k+d}$

▸**Page 74** lines 4 and 5 after (74) — 10 Jun 2011

inequalities ⇝ relations (twice)

▸**Page 81** line 4 of exercise 26 — 14 Jun 2011

Exhibit ⇝ Efficiently exhibit

Page 84 line 16 of exercise 35. — 21 Apr 2019

we get ⇝ in that order, we get

Page 85 line 2 of exercise 39 — 06 Nov 2013

N internal nodes ⇝ $N > 0$ internal nodes

Page 89 reordering of exercise 67 in the top several lines — 08 Jul 2011

The original part (b) ['If $n > 0$, let t^* ...'] becomes part (a); and the original part (a) ['Prove that, ...'] becomes part (b).

▸**Page 93** replacement for part (d) of exercise 110 — 20 Jul 2011

d) Suppose f is a pure majority function, namely a threshold function of the form (86) with $a = b = 0$. Then $s_1 \geq s_2 \geq \cdots \geq s_n$ implies that $w_1 \geq w_2 \geq \cdots \geq w_n$.

▸**Page 94** line 5 ——— 22 Jan 2012

$A_{n0}, A_{(n-1)1}, \ldots, A_{0n}$ ⇝ $A_{m0}, A_{(m-1)1}, \ldots, A_{0m}$

Page 96 line 6 ——— 20 Aug 2011

all vectors z. ⇝ all n-bit vectors z.

Page 100 line 8 ——— 10 Jun 2015

without ⇝ in which each step x_i is normal, without

Page 103 just before the displayed special chains ——— 08 Aug 2011

cannot obviously be shortened: ⇝ cannot be shortened in an obvious way:

Page 108 the line after (28) ——— 28 May 2014

needed to compute ⇝ that will compute

▸**Page 109** line 5 ——— 03 Jan 2014

$S_1 = \bar{z}_0 \wedge \bar{z}_1 \wedge z_2$ ⇝ $S_1 = z_0 \wedge \bar{z}_1 \wedge \bar{z}_2$

▸**Page 109** line −15 ——— 03 Jan 2014

exercise 79 ⇝ exercise 78

▸**Page 110** line 3 before (35) ——— 14 Jan 2013

Isvestiia ⇝ *Izvestiı͡a*

Page 114 replacement for (45) and the line above it ——— 05 May 2011

only 11 — a chain with fewer than 1.6 operations per output(!):

$$\begin{array}{lll} x_5 = x_1 \vee x_2, & \bar{d} = \;\; x_9 = x_6 \oplus x_8, & \bar{c} = x_{13} = \bar{x}_4 \wedge x_{10}, \\ x_6 = x_3 \oplus x_5, & \bar{f} = x_{10} = \bar{x}_5 \wedge x_8, & \bar{e} = x_{14} = x_4 \vee x_9, \\ x_7 = \bar{x}_2 \wedge x_6, & \bar{b} = x_{11} = x_2 \wedge \bar{x}_9, & g = x_{15} = x_6 \vee x_{11}. \\ x_8 = x_4 \vee x_7, & \bar{a} = x_{12} = \bar{x}_3 \wedge x_9, & \end{array} \tag{45}$$

This amazing chain, found by Corey Plover in 2011, chooses x_7 non-greedily.

▸**Page 117** in (56) ——— 03 Jan 2014

$\vee$ ⇝ $\wedge$ (twice)

▸**Page 117** in the line after (56) ——— 13 Jul 2016

396 gates ⇝ 394 gates

Page 118 line 9 ——— 18 Jun 2018

humungous ⇝ humongous

Page 120 line −16 ——— 11 Nov 2021

Section 7.4.1 ⇝ Section 7.4.1.2

Page 125 replacement for exercise 4 ——— 23 Apr 2021

4. [*M28*] Prove that the minimum depth and formula length of a Boolean function satisfy $\lg L(f) < D(f) < \alpha \lg L(f)$ when $L(f) > 1$, where $\alpha = 1/\lg \rho \approx 2.464965$ is related to the "plastic constant" ρ of Eq. 7.1.4–(90). *Hint:* If f contains a subformula g, we have $f = (g?\ f_1\colon f_0)$ for suitable f_1 and f_0.

Page 126 replacement for exercise 15 ——— 01 Dec 2011

15. [*28*] Find short-as-possible ways to evaluate the following Boolean functions using minimum memory: (a) $S_1(x_1, x_2, x_3)$; (b) $S_2(x_1, x_2, x_3, x_4)$; (c) $S_1(x_1, x_2, x_3, x_4)$; (d) the function in (18).

Page 128 rating of exercise 42 ——— 07 Sep 2011

25 ⇝ *30*

Page 129 line 3 of exercise 58 ——— 01 Nov 2011

Russian ⇝ USSR All-Union

Page 131 rating of exercise 76 ——— 16 Oct 2011

M25 ⇝ *M26*

▸**Page 131** line 8 of exercise 76 ——— 16 Oct 2011

$O(n/\log n)^2$ ⇝ $O(n^2)$

Page 131 line 10 of exercise 76 ——— 16 Oct 2011

length 2^m. ⇝ length 2^m, and the one-bit quantity $([l \le i] \oplus [i \le j])$ is ANDed with each component of $x \oplus y$.

▸**Page 131** line 3 of exercise 78 ——— 15 Nov 2013

$2\|S\| - 2$ ⇝ $2|S| - 2$

Page 132 rating of exercise 80 ——— 24 Oct 2011

M27 ⇝ *M29*

Page 133 line −2 of exercise 86 ——— 01 Dec 2011

$r^2q \ge 2^{r-1}$. ⇝ $r^2q \ge 2^{r+1}$. *Hint:* Now use the first formula.

Page 133 rating of exercise 87 ——— 30 Nov 2011

M20 ⇝ *M22*

Page 134 line 20 ——— 17 Nov 2011

technique is a trick ⇝ *technique* is a mature trick

Page 140 in (36), (39), (42), (43) ——— 31 Jul 2019

[change ‘remove’ to ‘replace by zero’]

Page 140 replacement for line −3 ——— 28 Dec 2014

$$\rho(2y+1) = 0, \qquad \rho(2y) = \rho(y) + 1 \tag{44}$$

Page 153 replacement for lines 7–11 ——— 30 Jun 2016

we can also do bytewise comparison in the cases where $c > 128$. Here's a 6-step formula that sets $t_j \leftarrow 128[x_j < y_j]$ for each byte position j in the test word t:

$$t \;\leftarrow\; \big((\bar{x} \,\&\, y) + (((\bar{x} \oplus y) \gg 1) \,\&\, \bar{h})\big) \,\&\, h. \tag{93}$$

(See exercise 96. This trick was discovered in 2013 by Norbert Juffa.) Notice that $((\bar{x} \oplus y) \gg 1) \,\&\, \bar{h}$ is the same as the quantity $((\bar{x} \oplus y) \,\&\, \bar{l}) \gg 1$ that we saw in (88).

▸**Page 157** replacement for the display in the proof of Theorem R — 04 Aug 2011

$$\{2^k,\, 2^{k+2^d},\, 2^{k+2\cdot 2^d},\, \ldots,\, 2^{k+(d-1)2^d}\} \cap V_{d0r} \;=\; \emptyset.$$

Page 169 append to the caption of Fig. 14 — 28 Jun 2014

[See H. A. Schwarz, *Crelle* **75** (1873), page 318 and Tafel II following page 348.]

Page 171 the line before Fig. 15 — 25 Jan 2011

sides: ⤳ sides (Fig. 15(b)):

Page 172 the line before (158) — 09 Dec 2015

that lead from ⤳ that begin with

▸**Page 173** line 1 after Fig. 16 — 15 Apr 2011

at least two ⤳ one, two, or three

Page 186 line 2 of exercise 25 — 01 Feb 2019

printed on 250 sheets ⤳ printed in English on 250 sheets

Page 189 rating of exercise 71 — 07 Sep 2011

17 ⤳ *20*

▸**Page 194** line 3 of exercise 127 — 24 Feb 2011

boardword ⤳ broadword

Page 197 replacement for line 6 of exercise 173 — 03 May 2016

$$X^D = \bigcap\{Y \mid Y \supseteq X \text{ and } Y \text{ is closed}\}; \quad X^L = \bigcup\{Y \mid Y \subseteq X \text{ and } Y \text{ is open}\}.$$

Page 198 line 9 of exercise 176 — 25 Apr 2012

$u' \overset{*}{\text{—}} v'$ in G' ⤳ $u' \overset{*}{\text{—}} v'$ in G' (that is, they are in the same component).

Page 198 line 3 of exercise 184 — 20 Mar 2012

order. ⤳ order. *Hint:* There is a simple answer.

Page 198 rating of exercise 185 — 21 Mar 2012

22 ⤳ *23*

Page 199 line 2 of exercise 189 — 23 Mar 2012

rotate it ⤳ rotate it counterclockwise

Page 200 line 4 of exercise 196 — 30 Mar 2012

$\alpha_j = 2^7 + x_{l-j}$ ⤳ $\alpha_j \leftarrow 2^7 + x_{l-j}$

Page 201 in exercise 203 — 13 Feb 2018

line 1: Suppose ⤳ (József Dallos, 2018.) Suppose
line 6: `XOR t,s,t; AND t,t,b` ⤳ `MOR t,b,s`

Page 202 new exercise — 16 Nov 2013

▸**219.** [*20*] What does this hack do? “While $x\&(x{+}1) \neq 0$, set $x \leftarrow x-((x\&(x{+}1))\gg 1)$.”

Page 207 line −6 — 12 Feb 2011

is best possible ⤳ is the best possible

Page 208 top — 20 Aug 2019

[I've changed the wording so as not to imply that (12) contains three BDDs]

Page 208 line −5 — 08 Jun 2012

So we set $x_1 \leftarrow 0$ with probability 13/18, and $\leadsto$ So, with probability 13/18, we set $x_1 \leftarrow 0$ and

Page 209 line 2 of step B3 — 01 Jun 2012

Then if $h \neq 0$, compute $\leadsto$ Then if $h \neq 0$, do the following: Compute

Page 214 append a sentence to the paragraph preceding Theorem M 08 Jun 2012

Kenneth McMillan has discovered an interesting upper bound that holds whenever we can formulate a computation using this general setup.

Page 225 new notation in line 4 — 16 Dec 2015

$v = f_v \leadsto v = f_o$

Note: Similarly, I want to replace $f_v \leadsto f_o$, $g_v \leadsto g_o$, $h_v \leadsto h_o$ **everywhere except in (72) and in the answer to exercise 7.1.4–93; this is a total of 104 changes!**

Page 235 line 1 before (86) — 08 Jun 2012

(84) implies $\leadsto$ (85) implies

Page 236 above and below (90) — 23 Apr 2021

$\chi \leadsto \rho$ (6 times)

▸**Page 243** line 5 — 18 May 2012

$\frac{1}{2}B(f)^2 \leadsto \frac{1}{8}B(f)^2$.

Page 244 line 15 — 20 Apr 2019

consists of applying $\leadsto$ applies

▸**Page 248** line −3 — 20 May 2012

$3 \cdot 2^{n/2} \leadsto 3 \cdot 2^{(n+1)/2}$

Page 249 line 5 before (122) — 20 Aug 2019

node branches to a $\leadsto$ node's HI points to a

Page 252 line −22 — 23 Aug 2016

unlike to exist $\leadsto$ unlike to be helpful

Page 253 caption to Fig. 28 — 17 Oct 2018

omitted from the ZDD $\leadsto$ omitted from this ZDD diagram

▸**Page 256** line 2 of Theorem S — 15 Jan 2015

$P \cup (x_1 \sqcup Q) \leadsto P \cup (e_1 \sqcup Q)$

▸**Page 256** line −18 — 17 Oct 2018

one subtree $\leadsto$ one subforest

▸**Page 256** line −10 ———— 21 May 2012

complete different ⇝ completely different

Page 257 line −18 ———— 11 Jan 2021

Patrician trees ⇝ Patricia trees

Page 258 line 8 ———— 26 Mar 2020

Centre d'Etudes Recherche Operationelle ⇝ *Centre d'Etudes de Recherche Operationelle*

Page 263 line 3 of exercise 76 ———— 06 Aug 2013

of such sets ⇝ of subsets of $\{0, 1, \ldots, n-1\}$

Page 263 rating of exercise 77 ———— 25 Aug 2012

M30 ⇝ *M35*

Page 266 rating of exercise 111 ———— 15 Sep 2012

M21 ⇝ *M22*

▸**Page 267** last line of exercise 130 ———— 01 Nov 2011

$\Omega(2^{m/3}/\sqrt{m}\,)$ ⇝ $\Omega(2^{m/3-O(\sqrt{m})})$

Page 267 last line of exercise 131 ———— 19 Oct 2011

Prove, however, that ⇝ Prove that

Page 271 new ratings for exercises ———— 29 Apr 2013

exercise 172, *M27* ⇝ *M28*
exercise 173, *HM28* ⇝ *HM33*

Page 272 append a hint to exercise 187 ———— 30 Aug 2019

[*Hint:* Just as a BDD is obtained from a QDD by removing nodes with HI = LO, a ZDD is obtained by removing nodes with HI = ⊥.]

Page 275 exercise 225 ———— 30 Jul 2017

Design an algorithm that will produce ⇝ Design an algorithm SIMPATH that produces

▸**Page 278** line 3 of exercise 253 ———— 04 Oct 2011

$\mathrm{PI}(f_2) \setminus A$ ⇝ $\mathrm{PI}(f_1) \setminus A$

Page 283 some clarifications ———— 13 Aug 2014

line 3 after (4): rotating disk ⇝ rotating mechanical disk
line 4 after (4): four sensors ⇝ four fixed optical sensors

Page 285 line 6 ———— 19 Aug 2021

the "tiring irons." ⇝ "tarrying irons" or "tiring irons."

Page 286 line 8 ———— 29 Mar 2011

successively ⇝ starting with 00…0 and successively

Page 286 in Algorithm G — 17 Oct 2015

line 3 of the algorithm: maintaining a parity bit a_∞ $\rightsquigarrow$ maintaining an odd parity bit a_{-1} (as suggested by G. Rote)
displayed equation (14): $a_\infty = a_{n-1}$ $\rightsquigarrow$ $a_{-1} = 1 \oplus a_{n-1}$
in step G4: If a_∞ ... minimum $\rightsquigarrow$ Let $j \geq 0$ be minimum
in steps G1, G3, and the line following the algorithm: $a_\infty \rightsquigarrow a_{-1}$ [four changes]

▸**Page 286** corrections to that amendment — 23 Jun 2020

line 3 of Algorithm G: odd parity bit $\rightsquigarrow$ even parity bit
displayed equation (14): $a_{-1} = 1 \oplus a_{n-1}$ $\rightsquigarrow$ $a_{-1} = a_{n-1}$
step G1 becomes simply this: Set $a_j \leftarrow 0$ for $-1 \leq j < n$.

Page 288 four lines before (20) — 27 Jul 2011

The *Walsh transform* $\rightsquigarrow$ The *Walsh transform*

▸**Page 291** line 2 of step W1 — 17 Oct 2015

$m' \mathbin{\&} (z-m) \mathbin{\&} \bar{m} \neq 0 \rightsquigarrow m' \mathbin{\&} (z-m) \mathbin{\&} \bar{z} \neq 0$

▸**Page 291** line 2 of step W1 — 15 May 2013

$m' = 2^5 m \rightsquigarrow m' = 2^4 m$

Page 292 lines -8 *and* -7 — 17 Oct 2015

of unnecessary work $\rightsquigarrow$ of work

▸**Page 304** lines 2–4 of Algorithm D — 18 Sep 2014

n zeros. (The two cycles may be identical, but they must be $\rightsquigarrow$ $n \geq 2$ zeros. (The two cycles must be identical; but they are to be

Page 304 in (58) — 09 Dec 2015

$0 \leq r \leq m \rightsquigarrow 1 \leq r \leq m$

Page 306 line 1 after (59) — 20 Dec 2013

for $L_m(n)$ using exercise 4.5.3–28(a), and $\rightsquigarrow$ for $L_m(n)$, using the Möbius function and exercise 4.5.3–28(a);

Page 308 rating of exercise 5 — 25 Dec 2013

20 $\rightsquigarrow$ *22*

Page 308 notation change in exercise 12(a) — 12 May 2016

$s_j \rightsquigarrow s_t$

Page 309 notation change in exercise 13 — 12 May 2016

$s_j \rightsquigarrow s_t$ (twice)

▸**Page 311** line 4 of exercise 28 — 17 Jan 2014

$\alpha_{kj} \rightsquigarrow a_{kj}$

▸**Page 312** line 2 of exercise 40 — 04 Feb 2014

$2^{5j} - 1 \rightsquigarrow 2^{5j+5} - 1$

Page 313 rating of exercise 43 —— 08 Nov 2011

47 $\leadsto$ *41*

Page 315 rewording of exercise 74 —— 27 May 2013

74. [*HM25*] (Savage and Winkler.) Prove that adjacent vertices of the n-cube cannot be separated by more than $O(2^n/\sqrt{n}\,)$ positions in a monotonic Gray code.

Page 317 rating of exercise 95 —— 24 Apr 2014

M23 $\leadsto$ *M24*

Page 321 line 4 of answer 133 —— 11 Nov 2021

Section 7.4.1 $\leadsto$ Section 7.4.1.3

Page 324 line 23 —— 17 Jul 2014

A computer program for solving alphametics was published by John Beidler in $\leadsto$ Early computer programs for solving alphametics were published by R. M. Burstall, *Comp. J.* **12** (1969), 48–51; John Beidler,

Page 326 line -15 —— 27 Aug 2011

saying that $j\alpha$ is "the image $\leadsto$ saying that "$j\alpha$ is the image

▶**Page 327** line 9 after (12) —— 07 Mar 2013

Computational Methods $\leadsto$ *Computational Problems*

Page 334 line 21 —— 27 May 2017

M. C. Er [*Comp. J.* **30** (1987), 282] $\leadsto$ J. S. Rohl [*Inf. Proc. Letters* **17** (1983), 231–234]

▶**Page 336** line 25 —— 05 Nov 2014

because of (32) $\leadsto$ because of (33)

Page 336 step C4 —— 12 May 2013

$k > 1$. $\leadsto$ $k > 1$; otherwise stop.

Page 342 line 17 —— 03 Dec 2021

total ordering $\leadsto$ total ordering, also called a linear ordering

Page 351 new rating for exercise 71 —— 31 Jul 2020

48 $\leadsto$ *41*

Page 353 rating of exercise 101 —— 18 Feb 2015

21 $\leadsto$ *28*

Page 357 after Table 1 —— 14 Mar 2015

line 2: positions $\leadsto$ positions (subscripts)
line 4: leftmost position $\leadsto$ leftmost position in (13)

Page 358 line -13 —— 30 May 2017

This algorithm generates $\leadsto$ This algorithm visits

▸**Page 359** line 18 ——— 24 Mar 2011

$t < n$ ⤳ $0 < t < n$

Page 361 line 19 ——— 30 May 2017

This algorithm generates ⤳ This algorithm visits

Page 363 line 11 ——— 21 Apr 2011

$n > t > 1$. ⤳ $n \geq t > 1$. An auxiliary variable c_{t+1} is used.

Page 363 last line of Algorithm R ——— 01 Apr 2011

if $j \leq t$. ▮ ⤳ if $j \leq t$. Otherwise the algorithm terminates. ▮

Page 365 line 13 ——— 16 Apr 2015

genlex methods ⤳ *genlex* methods

Page 381 new rating for exercise 23 ——— 05 Aug 2015

M23 ⤳ *M24*

Page 384 new rating for exercise 56 ——— 18 Mar 2015

M49 ⤳ *M41*

▸**Page 387** line −2 ——— 25 Jul 2018

if and only ⤳ if and only if

▸**Page 389** exercise 111 ——— 27 Jul 2018

[this exercise number shouldn't stick out into the left margin]

Page 393 line −4 ——— 12 Mar 2021

Philosophical Mag. **5** ⤳ *Philosophical Mag.* (4) **5**

Page 399 line 11 ——— 19 Apr 2011

$O(n^{1/2})$ steps. ⤳ $O(n^{1/2})$ steps. [See *Handbook of Combinatorics* **2** (MIT Press, 1995), 1068–1069.]

Page 414 line 6 of exercise 58 ——— 10 Jun 2018

$(y_1 + \cdots + y_n)/n \geq (y_1 \ldots y_n)^{1/n}$ in the special case $m = n$, ⤳ $(y_1 + \cdots + y_n)/n \geq (y_1 \ldots y_n)^{1/n}$, "the arithmetic mean dominates the geometric mean," in the special case $m = n$,

Page 418 line −14 ——— 11 Jun 2017

exercises 26–31 ⤳ exercises 26–31 and 33

▸**Page 418** line −3 ——— 28 Jul 2016

$\Theta(n/\log n)^n$ ⤳ $\bigl(\Theta(n/\ln n)\bigr)^n$

Page 423 line −5 ——— 20 Jan 2018

(29) ⤳ (26)

Page 436 rating of exercise 34 ——— 06 Jul 2021

14 ⤳ *34*

Page 443 lines 3 and 4 after (5) — 20 Apr 2019

consists of scanning from the right ... and replacing $\rightsquigarrow$ scans from the right ..., and replaces

Page 448 in the introduction to Algorithm L — 09 Dec 2015

$l_k \rightsquigarrow l_j$ (twice)
$r_k \rightsquigarrow r_j$ (twice)
kth node $\rightsquigarrow$ jth node (twice)

▸**Page 449** the rightmost column of Table 3 — 23 Apr 2019

[The 'dual' forests in this table were almost completely wrong, in early printings! They should actually be the following, from top to bottom: ; ; ; ; ; ; ; ; ; ; ; ; ; .]

Page 456 lines −3 and −2 — 29 Nov 2013

is node L_0; ... L_{2k} $\rightsquigarrow$ is node number L_0; the left and right children of branch node $2k-1$ are respectively the nodes that are numbered L_{2k-1} and L_{2k}

Page 457 line 9 — 29 Nov 2013

to R2. $\rightsquigarrow$ to R2. (Here L_k corresponds to x in (33).)

Page 461 new paragraph before 'Oriented trees and forests' — 20 Dec 2019

P. Gregor, O. Mička, and T. Mütze have found a nice way to reorder the rows of the Christmas tree pattern so that we obtain a Gray code for all binary n-tuples by reading alternately left-to-right and right-to-left. (See arXiv:1912.01566 [math.CO] (2019), 33 pages.)

Page 462 line 16 — 02 Feb 2012

root. $\rightsquigarrow$ root. The algorithm sets $p_0 \leftarrow -1$.

Page 464 line 2 — 29 Dec 2017

then $\rightsquigarrow$ then we arrange everything in memory so that

▸**Page 472** line 8 of exercise 20 — 12 Aug 2020

$1 \le k \le N \rightsquigarrow 0 \le k \le N$

▸**Page 472** line 4 of exercise 21 — 23 Apr 2019

and the valid $\rightsquigarrow$ the valid

Page 474 caption to Fig. 61 — 23 Apr 2019

(a) node depths and (b) descendant counts $\rightsquigarrow$ (a) node depths $c_1c_2c_3c_4$ and (b) descendant counts $s_1s_2s_3s_4$

Page 475 caption to Fig. 62 — 23 Apr 2019

node depths $\rightsquigarrow$ node depths $c_1c_2c_3c_4$

Page 475 replacement for exercise 30 — 13 Aug 2020

30. [*M33*] The *canopy* of a forest is the bit string $f_1 \ldots f_{n-1}$ defined by

$$f_j \;=\; [\text{node } j \text{ in preorder is not a leaf}].$$

a) If F has canopy $f_1 \ldots f_{n-1}$, what is the canopy of F^D? (See exercises 26 and 27.)
b) How many forests have the canopy 1010110111111000010101000101100?
c) Prove that $f_j = [d_j = 0]$, for $1 \le j < n$, in the notation of (6).
d) Two elements of a lattice are called *complementary* if their greatest lower bound is the bottom element while their least upper bound is the top element. Show that F and F' are complementary in the Tamari lattice if and only if their canopies are complementary, in the sense that $f'_1 \ldots f'_{n-1} = \bar{f}_1 \ldots \bar{f}_{n-1}$.

▸**Page 475** line 1 of exercise 31 — 29 Jun 2011

height $n-1$ ⇝ height n

Page 479 in exercise 71 — 09 Dec 2015

line 1 and line 3: M_{tn} ⇝ $M_{t,n}$
line 3: M_{1n} ⇝ $M_{1,n}$

Page 480 new rating for exercise 85 — 09 Dec 2015

M35 ⇝ *M30*

Page 482 new copy after line 5 of exercise 104 — 15 Jan 2019

[This important matrix is called the *Laplacian* of G; see answer 2.3.4.2–19.]

Page 490 line 15 — 27 Apr 2017

his commentary ⇝ his commentary *Ḥakhmoni*

Page 490 lines 23–25 — 27 Apr 2017

Although he knew … generating them. ⇝ He knew how many permutations there were supposed to be, and sketched a “rolling” method that would extend his list to the 5040 that arise with seven letters.

Page 492 line 16 — 21 Apr 2019

alternatively alive, dead ⇝ alternately alive, dead

Page 495 lines -13 *and* -12 — 09 Jun 2012

respectively simplicity, rank, mercy, and sovereignty. ⇝ simplicity, rank, mercy, and sovereignty, in that order.

Page 500 in (20) — 19 Aug 2019

[insert an ellipsis ‘…’ at the beginning of each line of this display]

Page 501 line -6 — 01 Mar 2014

dum-dum dum-dum dum-diddy ⇝ dum, dum dum, dum dum-diddy

Page 502 lines -9 through -6 — 24 Jan 2012

Those pages … by mistake. ⇝ Bernoulli didn’t actually intend to publish those pages in this now-famous book; but the proofreader who found them among his notes decided to include the full details, in order “to gratify curiosity.”

Page 504 lines 2 and 3 after (26) ——— 19 Aug 2021

Arima's book *Shūki Sanpō* came out in 1769. Problem 56 of that book ⇝ Arima's five-volume *Shūki Sanpō* came out in 1769. The problem on pages 32–35 of volume 2

▸**Page 509** line 7 ——— 27 Jun 2012

Papers, Volume 4 ⇝ *Papers*, Volume 9

Page 509 line 2 before (33) ——— 12 Mar 2021

Philosophical Magazine **18** ⇝ *Philosophical Magazine* (4) **18**

Page 514 line 2 of the first answer 2 ——— 13 Sep 2011

its answer. ⇝ its answer, assuming that he or she is suitably sagacious.

Page 514 new format for Perelman references in answer 3 ——— 12 Aug 2019

arXiv:math/0211159 [math.DG] (2002), 39 pages; 0303109 and 0307245 [math.DG] (2003), 22+7 pages.

Page 515 line 12 ——— 24 Apr 2012

$2^{30}+1$. ⇝ $2^{30}+1$, because $2^{49} \perp 2^{60}-1$.

Page 515 line −4 of answer 8 ——— 31 May 2017

nonprimary columns ⇝ nonprimary items

Page 515 line −3 of answer 8 ——— 24 Dec 2017

7.2.2.1–00 [or in the 10th printing, 7.2.2.1–26] ⇝ 7.2.2.1–27

Page 515 line −2 of answer 8 ——— 19 Aug 2015

Skolem's variant of the problem ⇝ the problem of exercise 2

Page 516 last two lines of answer 10 ——— 05 Jun 2018

See also an article ... Andersen. ⇝ See also Lars D. Andersen's excellent history of latin squares in *Combinatorics: Ancient and Modern* (Oxford University Press, 2013), Chapter 11.

Page 517 replacement for answer 17 ——— 31 May 2017

17. (a) Let there be $3n$ vertices r_j, c_j, v_j for $0 \le j < n$, and n^2 hyperedges; hyperedge (i,j) consists of the three vertices $\{r_i, c_j, v_l\}$, where $l = L_{ij}$, for $0 \le i,j < n$.

(b) Let there be $4n^2$ vertices r_{ij}, c_{ij}, x_{ij}, y_{ij} for $0 \le i,j < n$, and $n^3 - n^2 + n$ hyperedges; hyperedge (i,j,k) consists of the four vertices $\{r_{ik}, c_{jk}, x_{ij}, y_{lk}\}$, where $l = L_{ij}$, for $0 \le i,j,k < n$ and ($i = k$ or $j > 0$).

Page 517 line −3 ——— 23 May 2011

L is a latin square ⇝ L with entries in $\{1, 2, \ldots, n\}$ is a latin square

Page 521 in answer 47 ——— 05 Sep 2020

[The fact that $K_{3,3}$ is nonplanar ⇝ [W. Blaschke and G. Bol called $K_{3,3}$ "Thomsen's three corners" in *Geometrie der Gewebe* (1937). The fact that it is nonplanar

Page 521 in answer 49 ——— 18 Nov 2015

which is $\overline{C}_6$ ⇝ which is $\overline{C}_6$ and $K_2 \,\square\, K_3$

Page 522 new paragraph at answer of answer 54 ——— 18 Jul 2012

[This digraph was first considered by Darryl Francis, Philip Cohen, and A. Ross Eckler in *Word Ways* **19** (1976), 241; **20** (1977), 8.]

▸**Page 522** line 2 of answer 63 ——— 23 May 2011

$1 \le k < t \rightsquigarrow 1 \le k \le t$

▸**Page 523** line 4 of answer 71 ——— 23 May 2011

`ASIZE IS 24` $\rightsquigarrow$ `ASIZE IS 20`

Page 524 bottom line ——— 17 Sep 2012

 $\rightsquigarrow$  (called the "bull").

Page 525 replacement for answer 88 ——— 23 Dec 2015

88. One cycle that doesn't use the center vertex, plus $n(n-1)$ cycles that do (namely, one for every ordered pair of distinct vertices on the rim). Just $n+1$ are induced.

Page 526 line 2 of answer 90 ——— 23 May 2011

cographs by (47) $\rightsquigarrow$ cographs by (39)

Page 526 line 3 of answer 90 ——— 23 Dec 2015

W_5 is a cograph, but not W_6 $\rightsquigarrow$ W_4 is a cograph, but not W_5

▸**Page 526** last line of answer 97 ——— 20 May 2011

$K_m \otimes K_n = \overline{K_m \,\square\, K_n} = L(K_{m,n}) \rightsquigarrow K_m \,\square\, K_n = \overline{K_m \otimes K_n} = L(K_{m,n})$

▸**Page 527** line 5 of answer 101 ——— 23 May 2011

shortest cycle $\rightsquigarrow$ shortest odd cycle

Page 527 line 15 of answer 105 ——— 26 May 2011

$u = C_t$, and $v = c_t$. $\rightsquigarrow$ and $u = C_t$.

▸**Page 528** line 4 ——— 23 May 2011

$0 \le a_j \rightsquigarrow 0 \ge a_j$

▸**Page 528** lines 5 and 6 of answer 108 ——— 23 May 2011

In step H4, omit "$c_j \leftarrow c_j - 1$"; change "$n — m$" to "$m \longrightarrow n$"; $\rightsquigarrow$ In step H4, change "Set ... and set" to "If $j > 0$, set $m \leftarrow c_t$, create the arc $m \longrightarrow n$, and set";

Page 528 line 4 of answer 110 ——— 23 May 2011

$d = 2k \rightsquigarrow d = 2k > 0$

Page 529 line 5 of answer 114 ——— 23 May 2011

bipartite graph $\rightsquigarrow$ bipartite multigraph

▸**Page 529** line 4 of answer 122 ——— 29 May 2011

other three colors $\rightsquigarrow$ other three vertices

▸**Page 530** last line of answer 126 ——— 23 May 2011

1978 $\rightsquigarrow$ 1979

▸**Page 530** line 3 of answer 131 ——— 11 Jan 2014

68(a) $\rightsquigarrow$ 68(b)

▸**Page 531** in answer 135 ——— 23 May 2011

line 1: $\omega = e^{2\pi i/6}$ ⇝ $\omega = e^{2\pi i/12}$
line 5: TD^mT^- ⇝ $TD^{2m}T^-$

▸**Page 532** in answer 138 ——— 27 Oct 2015

The six neighbors ... degree 8. ⇝ The six neighbors of (u, v) are then $((u \pm 1) \bmod 2, (v \pm 6) \bmod 10)$, $((u \pm 1) \bmod 2, (v \pm 1) \bmod 10)$, $(u, (v \pm 1) \bmod 10)$. One consequence is that 3D space can be "tiled" with interesting 20-element subsets of a $3 \times 3 \times 3$ cube.

Page 533 replacement for line 1 of answer 140 ——— 23 Dec 2015

140. (a) $\#(K_3{:}W_n) = n$ and $\#(P_3{:}W_n) = \binom{n}{2}$ for $n \geq 4$; also $\#(\overline{K_3}{:}W_7) = 7$.

Page 533 line 1 of answer 142 ——— 24 May 2011

the previous answer ⇝ answer 140

Page 534 last line of answer 145 ——— 15 Jul 2013

152.] ⇝ 152. See also the analysis by Alonso and Reingold, *Information Processing Letters* **113** (2013), 495–497.]

Page 535 line line of answer 7 ——— 19 Apr 2020

in Section 7.2.3 ⇝ in Section 7.2.2.1

Page 536 line 8 ——— 28 Nov 2013

This expansion is a disguised form of the ⇝ The latter, with k $*$'s, is a k-variable

▸**Page 539** line −6 of answer 30 ——— 13 Oct 2013

essentially proportional to mn (for step P1) plus the ⇝ at most proportional to mn (for step P1) plus n times the

Page 539 lines −10 and −9 of answer 31 ——— 21 Dec 2013

first defined by ... see ⇝ was first defined (under the name "syllogistic result") by Archie Blake on page 25 of his Ph.D. dissertation, *Canonical Expressions in Boolean Algebra* (University of Chicago, 1937); see

▸**Page 540** lines 2 and 3 ——— 19 Jun 2011

the generalized consensus if and only if this subcube is contained in $c_1 \cup \cdots \cup c_m$. ⇝ the only subcube that could possibly be a generalized consensus.

Page 540 added remark for end of answer 33 ——— 20 Jun 2011

(See, for example, Eq. 1.2.6–(24).)

▸**Page 541** line 3 of answer 34(c) ——— 16 Oct 2013

$\lg\lg\lg n$ ⇝ $\ln\lg\lg n$

Page 542 replacement for lines 5–8 of answer 39 ——— 09 Dec 2015

$$c_{pq} = (y^{\bar{p}\circ\bar{q}} \vee l^p \vee r^q)$$

and x^p denotes $[x = p]$, so that $x^0 = \bar{x}$ and $x^1 = x$. These clauses state in effect that $y = l \circ r$; for example, if $\circ$ is $\wedge$, the four clauses are $(y \vee \bar{l} \vee \bar{r}) \wedge (\bar{y} \vee \bar{l} \vee r) \wedge (\bar{y} \vee l \vee \bar{r}) \wedge (\bar{y} \vee l \vee r)$. Finally, add one more clause, $(y_1 \vee y_1 \vee y_1)$, to force $f = 1$.

Page 542 line 9 of answer 41 ——— 11 Apr 2014

The total $\rightsquigarrow$ (Only the first and last of these four clauses are actually needed.) The total

Page 544 replacement for answer 54 ——— 09 Dec 2015

54. If x and $\bar{x}$ are both in S, then $u \in S \implies \bar{u} \in S$, because the existence of paths from x to u and u to x implies the existence of paths from $\bar{u}$ to $\bar{x}$ and $\bar{x}$ to $\bar{u}$. Similarly, $\bar{u} \in S \implies u \in S$.

Page 544 append new sentence to the last line ——— 22 Nov 2015

(The number of t variables can be reduced, as in exercise 7.2.2.2–12.)

▸**Page 545** line 3 ——— 02 Jan 2012

5.24 $\rightsquigarrow$ 5.2.4

Page 546 replacement for answer 67(a) ——— 08 Jul 2011

67. (a) A black Y in t forces a black Y in t^*, because adjacent black stones a — b — c in t yield two adjacent black stones in t^*. Similarly, a black Y in t^* forces a black Y in t.

Page 547 line −5 ——— 02 Jan 2012

$x = y \rightsquigarrow y = x$

Page 548 lines 16–18 of answer 77 ——— 10 Jul 2011

Since $d(w,v)$ ... has rank 1. $\rightsquigarrow$ The vertex $x = \langle svw \rangle$ has rank 1 by (c). If $x = v$, then u has rank 2 unless w has rank 0.

Page 551 new sentence at end of answer 88 ——— 14 Dec 2013

(Actually $O(t)$ delay suffices, as observed by Omid Etesami, if we rearrange clusters having the same d, since those can each be done in ≤ 9 units.)

▸**Page 552** line 1 of answer 102 ——— 17 Jul 2011

(1) and (2) $\rightsquigarrow$ (11) and (12)

Page 554 line 2 of answer 111 ——— 22 Jul 2011

set $\rightsquigarrow$ choose

Page 555 line −4 of answer 111 ——— 22 Jan 2012

Exercise 70 tells us that $\rightsquigarrow$ Since f_k is self-dual we have

Page 555 line 1 of answer 112(c) ——— 14 Dec 2013

over all $2^m \rightsquigarrow$ over all $n = 2^m$

Page 556 lines 4–6 ——— 14 Dec 2013

For example, ... hence $\rightsquigarrow$ For example, if $k \ge 0$ and $f(x_1, \ldots, x_m) = x_1 \oplus \cdots \oplus x_k \oplus [k\,\text{even}]$, we find $F_\alpha = 0$ for all α except that $F_{1^k 0^{m-k}} = 2^{m-1}$. The multilinear representation of that function has leading term $\lceil 2^{k-1} \rceil x_1 \ldots x_k$; hence

▸**Page 559** line 2 of answer 127 ——— 31 Jul 2011

x_j. $\rightsquigarrow$ x_j or $\bar{x}_j$.

Page 560 in lines 1 and 2 of answer 132(b) ——— 08 Aug 2011

Given $y = y_1 \ldots y_n$ ⇝ Given $y_0,\ y_1,\ \ldots,\ y_n$
$1 + h(y_1, y_3, \ldots, y_{n-1})$ ⇝ $1 + y_0 + h(y_1, y_3, \ldots, y_{n-1})$
$x \cdot y \bmod 2$ ⇝ $(y_0 + x \cdot y) \bmod 2$

Page 561 replacement for end of answer 132 ——— 07 Feb 2021

$n = 9.]$ ⇝ $n = 9$. See K.-U. Schmidt, *J. Combinatorial Theory* **A164** (2019), 50–59, for asymptotically optimum results as $n \to \infty$ through odd values.]

Page 561 replacement for lines 3–10 of answer 4 ——— 23 Apr 2021

The hint follows when we let f_t be the formula of size $L(f) - L(g) - 1$ that arises when g is replaced by t. For $1 \le k < L(f)$ let g_k be a minimal subformula of size $\ge k$. Then $g_k?\ f_{k1}\colon f_{k0}$ is obtained from a tree that has g_k, f_{k1}, and f_{k0} on level 2.

Let $d_r = \max\{\, D(f) \mid L(f) = r \,\}$. Since the children of g_k appear on level 3 and have size $< k$, we have $d_r \le \min_{k=1}^{r-1} \max(3 + d_{k-1}, 2 + d_{r-k-1})$ for $r \ge 3$. By induction on r it follows that $d_r \le l$ when $r \le b_l$, where $b_l = l$ for $0 \le l \le 2$ and $b_l = b_{l-2}+b_{l-3}+2$ for $l \ge 3$. We also have $b_l + 2 = (8P_l + 18P_{l+1} + 11P_{l+2})/23 = c\rho^l + O(0.87^l)$ in terms of the Perrin numbers of exercise 7.1.4–15, where $c = (2{+}4\rho{+}3\rho^2)/(3{+}2\rho) \approx 2.224$. Hence $d_r < \alpha \lg r$ when $r > 1$. [See P. M. Spira, *Hawaii Int. Conf. Syst. Sci.* **4** (1971), 525–527; R. Brent, D. Kuck, and K. Maruyama, *IEEE* **C-22** (1973), 532–534. In *JACM* **23** (1976), 534–543, D. E. Muller and F. P. Preparata proved that $D(f) \le \beta \lg L(f) + O(1)$, where $\beta = 1/\lg z \approx 2.0807$, $z^4 = 2z + 1$. Is β optimum?]

Page 562 replacement for lines 2–8 of answer 15 ——— 01 Dec 2011

(a)	(b)	(c)	(d)
$x_1 \leftarrow x_1 \oplus x_2$,	$x_1 \leftarrow x_1 \oplus x_2$,	$x_1 \leftarrow x_1 \oplus x_2$,	$x_1 \leftarrow x_1 \oplus x_2$,
$x_1 \leftarrow x_1 \oplus x_3$,	$x_3 \leftarrow x_3 \oplus x_4$,	$x_2 \leftarrow x_2 \wedge \bar{x}_1$,	$x_2 \leftarrow x_2 \oplus x_3$,
$x_2 \leftarrow x_2 \wedge x_3$,	$x_1 \leftarrow x_1 \oplus x_3$,	$x_3 \leftarrow x_3 \oplus x_4$,	$x_2 \leftarrow x_2 \vee x_1$,
$x_1 \leftarrow x_1 \wedge \bar{x}_2$.	$x_2 \leftarrow x_2 \oplus x_4$,	$x_4 \leftarrow x_4 \wedge x_1$,	$x_1 \leftarrow x_1 \oplus x_4$,
	$x_3 \leftarrow x_3 \vee x_2$,	$x_2 \leftarrow \bar{x}_2 \wedge x_3$,	$x_1 \leftarrow x_1 \wedge x_3$,
	$x_3 \leftarrow x_3 \wedge \bar{x}_1$.	$x_2 \leftarrow x_2 \oplus x_1$,	$x_2 \leftarrow x_2 \wedge \bar{x}_1$,
		$x_2 \leftarrow x_2 \wedge \bar{x}_4$.	$x_2 \leftarrow x_2 \oplus x_4$.

▸**Page 563** in some copies of the third printing only ——— 13 Jan 2013

Pages 563 and 564 were erroneously replaced by duplicates of pages 565 and 566 when the third printing went to the presses. The correct pages can be found online:

`http://www-cs-faculty.stanford.edu/~knuth/4a-563,564.ps.gz`

Page 563 line 8 ——— 09 Jan 2014

performed. ⇝ performed. (See (14).)

Page 564 last line of answer 20 ——— 21 Jan 2011

is the complement of (20) ⇝ is equivalent to (20)

Page 564 replacement for lines 3–6 of answer 21 ——— 21 Jan 2011

hardest function (20) is still unknown, but David Stevenson has shown that $V(f) \le 17$:

$$\begin{aligned} g &= \mathrm{AND}\big(\mathrm{NAND}(w,x), \mathrm{NAND}(\bar{w},\bar{x})\big);\\ f &= \mathrm{OR}\big(\mathrm{AND}\big(\mathrm{NOT}(g), \mathrm{NAND}(w,\bar{z}), \mathrm{NAND}(y,z)\big),\\ &\qquad \mathrm{AND}\big(\mathrm{NOT}(\mathrm{NOT}(g)), \mathrm{NAND}(y,\bar{z}), \mathrm{NAND}(\bar{y},z)\big)\big). \end{aligned}$$

▸**Page 565** line 4 of answer 23 ———— 17 May 2014

Of course S_1 $\rightsquigarrow$ Of course S_5

▸**Page 565** answer 28 ———— 03 Jan 2014

lines 1 and 3: $\overline{\subset}$ $\rightsquigarrow$ $\overline{\supset}$
line 4: $1 \le j \le \lfloor n/2 \rfloor$ $\rightsquigarrow$ $0 \le j \le \lfloor n/2 \rfloor$

▸**Page 565** line 4 of answer 28 ———— 17 May 2014

$(x_1 + \cdots + x_{2j+1}$ $\rightsquigarrow$ $(-1)^j(x_1 + \cdots + x_{2j+1}$

▸**Page 565** lines 4 and following of answer 29 ———— 03 Jan 2014

Hence $a_{jd} \le 5 \ldots n = 10^7$.) $\rightsquigarrow$ Hence $a_{jd} \le 4$ when $d \ge l = \lceil \log_{3/2} n \rceil$. It follows that $a_{j(j+l+3)} = 0$ for $0 \le j \le \lg n$, giving total depth $\le 3(l + \lg n + 2)$. (The actual depth turns out to be exactly 101 when $n = 10^7$, 118 when $n = 10^8$, 133 when $n = 10^9$.)

Page 565 last line of answer 31 ———— 28 May 2014

325.] $\rightsquigarrow$ 325. Exercise 7.2.2.2–481 improves $5n$ to $4.5n$.]

▸**Page 568** line 7 of answer 42 ———— 07 Sep 2011

$l = 2$ $\rightsquigarrow$ $l = 3$

Page 572 replacement for answer 57 ———— 19 May 2011

57. The truth tables for x_5 through x_{15}, in hexadecimal notation, are respectively `0fff`, `3ccc`, `30c0`, `75d5`, `4919`, `7000`, `0606`, `4808`, `2000`, `5d5d`, `3ece`. So we get

$$1010 \mapsto 0, \quad 1011 \mapsto 7, \quad 1100 \mapsto 4, \quad 1101 \mapsto 5, \quad 1110 \mapsto 6, \quad 1111 \mapsto 7.$$

[Corey Plover, believing that it might be better to have a solution in which nondigits never masquerade as digits, has discovered a 12-step chain (with non-greedy x_7)

$$\begin{array}{lll} x_5 = x_1 \oplus x_2, & x_9 = x_2 \oplus x_3, & \bar{b} = x_{13} = x_2 \wedge \bar{x}_{11}, \\ x_6 = x_3 \wedge \bar{x}_4, & g = x_{10} = x_7 \vee x_9, & \bar{c} = x_{14} = x_7 \wedge x_9, \\ x_7 = x_1 \oplus x_6, & \bar{d} = x_{11} = x_8 \oplus x_{10}, & \bar{e} = x_{15} = x_4 \vee x_{12}, \\ x_8 = x_4 \vee x_7, & \bar{a} = x_{12} = \bar{x}_3 \wedge x_{11}, & \bar{f} = x_{16} = \bar{x}_5 \wedge x_8, \end{array}$$

for which $a, \ldots, g$ have the truth tables `b7ff`, `f9f0`, `dfe3`, `b6df`, `a2aa`, `8ff2`, `3efd`, and

$$1010 \mapsto \mathtt{A}, \quad 1011 \mapsto , \quad 1100 \mapsto , \quad 1101 \mapsto , \quad 1110 \mapsto , \quad 1111 \mapsto .$$

He has also shown that all 11-step solutions to (44) map the nondigits into either (0, 7, 4, 5, 6, 7), (0, 7, 6, 1, 9, 3), (0, 7, 6, 1, A, 3), (2, 3, 6, 7, 6, 7), (2, 1, 6, 7, 4, 5), or (2, 4, 6, 7, 4, 5).]

▸**Page 573** line 1 of answer 62 ———— 13 Sep 2011

$t(n,r)$ $\rightsquigarrow$ $c(n,r)$

▸**Page 574** line 2 of answer 67 ———— 24 Apr 2011

ANDN $\rightsquigarrow$ BUTNOT

▸**Page 574** line 3 of answer 69 ———— 09 Oct 2011

α_{110} $\rightsquigarrow$ α_{011}

▸**Page 576** line 4 of answer 76 ———— 16 Oct 2011

$O(n/\log n)$ $\rightsquigarrow$ $O(n)$

▸**Page 576** bottom line ——— 19 Oct 2011

$x_1 = x_1 \circ x_2$ ⤳ $x_{n+1} = x_1 \circ x_2$

▸**Page 577** replacement for the first 10 lines ——— 19 Oct 2011

Case 4.1: $k > n$. Then $k > n+1$. If $u_k = 1$, set $x_1 \leftarrow t_l$ and remove steps $n+1$, k, l, U_l. Otherwise set $x_2 \leftarrow t_{n+1}$; this forces $x_k = \bar{t}_l$, and we can remove $n+1$, k, l, U_k.

Case 4.2: $x_l = x_1 \circ x_m$. Then we must have $m = 2$; for if $m > 2$ we could set $x_2 \leftarrow t_{n+1}$, $x_m \leftarrow t_l$, and make x_{n+r} independent of x_1. Hence we may assume that $x_{n+1} = x_1 \wedge x_2$, $x_{n+2} = x_1 \vee x_2$. Setting $x_1 \leftarrow 0$ allows us to remove U_1 and U_{n+1}; setting $x_1 \leftarrow 1$ allows us to remove U_1 and U_{n+2}. Thus we're done unless $u_{n+1} = u_{n+2} = 1$.

If $x_p = \bar{x}_{n+1}$, set $x_1 \leftarrow 0$ and remove $n+1$, $n+2$, p, U_p; if $x_q = \bar{x}_{n+2}$, set $x_1 \leftarrow 1$ and remove $n+1$, $n+2$, q, U_q. Otherwise $x_p = x_{n+1} \circ x_u$ and $x_q = x_{n+2} \circ x_v$, where x_u and x_v do not depend on x_1 or x_2. But that's impossible; it would allow us to set $x_3, \ldots, x_n$ to make $x_u = t_p$, then $x_2 \leftarrow 1$ to make x_{n+r} independent of x_1.

▸**Page 577** lines 2 and 4 of answer 78 ——— 15 Nov 2013

$\|S\|$ ⤳ $|S|$ (twice)

▸**Page 578** line 5 of case 3 in answer 80 ——— 24 Oct 2011

$u_{j'} = 1$ ⤳ $u_{j'} \neq 1$

▸**Page 578** near the end of answer 80 ——— 22 May 2014

Delete the parenthesized remarks '(According to ... $n/2$.)' and insert a new paragraph at the end:

(A nice unpublished idea of L. Adleman, circa 1978, proves that $C(S_{\geq 2}) = 2n + O(\sqrt{n})$: Given m^2 elements x_{ij}, compute $c \vee r$, where $c = S_{\geq 2}(\bigvee_{i=1}^m x_{i1}, \ldots, \bigvee_{i=1}^m x_{im})$ and $r = S_{\geq 2}(\bigvee_{j=1}^m x_{1j}, \ldots, \bigvee_{j=1}^m x_{mj})$ each cost $< m^2 + 3m$ using just $\vee$ and $\wedge$.)

▸**Page 580** in answer 86 ——— 01 Nov 2011

line 3 of (e): one of the functions $\epsilon_i = \hat{x}_i \oplus (\hat{x}_{j(i)} \wedge x_{k(i)})$. ⤳ the union of the terms $\epsilon_i = \hat{x}_i \oplus (\hat{x}_{j(i)} \wedge \hat{x}_{k(i)})$ for the p AND steps.

line 4 of (f): in one of the terms $T = \delta_i = \hat{x}_i \oplus (\hat{x}_{j(i)} \vee x_{k(i)})$ ⤳ within the terms $T = \delta_i = \hat{x}_i \oplus (\hat{x}_{j(i)} \vee \hat{x}_{k(i)})$

also replace line 8 of (f) through line 3 of (g) by the following:

monochromatic. We will prove that each term T includes at most $2^{n-2-r}r^2$ such B, hence $2^{n-2-r}r^2 q \geq 2^{n-1}$.

We can compute $G^* = G_t$ from any given graph G by the following (inefficient) algorithm: Set $G_0 \leftarrow G$, $t \leftarrow 0$. If G_t has an r-family S with $|\Delta(S)| < 2$, set $t \leftarrow t+1$, $G_t \leftarrow \infty$, and stop. Otherwise, if $\Delta(S) = \{u, v\}$ and $u \not\!\!\frac{\quad}{} v$, set $t \leftarrow t+1$, $G_t \leftarrow (G_{t-1}$ plus the edge $u \frac{\quad}{} v)$ and repeat. Otherwise stop.

There are 2^{n-1-r} bipartite minterms B with monochromatic $\{u_j, v_j\}$ for $1 \leq j \leq r$ when $|\Delta(S)| < 2$. And when $\Delta(S) = \{u, v\}$ there are 2^{n-2-r} with monochromatic $\{u_j, v_j\}$ and bichromatic $\{u, v\}$. Hence

$$T = \lceil G^* \rceil \setminus \lceil G \rceil = (\lceil G_t \rceil \setminus \lceil G_{t-1} \rceil) \vee \cdots \vee (\lceil G_1 \rceil \setminus \lceil G_0 \rceil)$$

contains $2^{n-2-r}(t + [G^* = \infty])$ minterms B. And the algorithm stops with $t \leq (r-1)^2$.

(g) Exercise 84 tells us that $q < \binom{p}{2} + (p+1)\binom{n}{2}$. Thus we have either $2(r-1)^3 p \geq \binom{n}{3} - (r-1)^2(n-2)$ or $\binom{p}{2} + (p+1)\binom{n}{2} > 2^{r+1}/r^2$. Both lower bounds for p are

$$\geq \frac{1}{6}\left(\frac{n}{6 \lg n}\right)^3 \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right) \qquad \text{when} \qquad r = \left\lceil \lg\left(\frac{n^6}{746496 (\lg n)^4}\right) \right\rceil.$$

▶**Page 581** near the top ——— 12 Sep 2019

line 9 (Classes for $C(f) = 6$): 353 $\leadsto$ 354; 798686 $\leadsto$ 799946
line 10 (Classes for $C(f) = 7$): 99 $\leadsto$ 98; 169891 $\leadsto$ 168631
line 15: S. Amarel $\leadsto$ (These results for $C(f)$ were corrected and independently verified by E. Testa in 2019.) S. Amarel

▶**Page 583** line 8 ——— 14 Nov 2013

$x'(y \oplus y') \leadsto x' \otimes (y \oplus y')$

Page 583 lines −7 thru −3 of answer 10 ——— 23 Jan 2011

$1 \le j \le 7 \leadsto 0 \le j \le 7$ and also include the constant $\mathtt{q}_0 = \mathtt{8040201008040201}$

▶**Page 584** line 2 of answer 13 ——— 28 Nov 2011

$a \ne b \leadsto x \oplus y = x' \oplus y'$ and $a \ne b$

Page 589 improved answer to exercise 34(c) ——— 10 Mar 2015

Proof of (c) by Liviu Lalescu: $x \oplus y = (x-1) \oplus (y-1)$ if and only if $x \oplus (x-1) = y \oplus (y-1)$; and $x \oplus (x-1) = 2^{1+\rho x} - 1$ by Eq. (41).

Page 589 append new paragraph to answer 38 ——— 13 Feb 2018

[But J. Dallos needs only $5v$! Do '`MOR y,a,x; SRU y,y,1; MOR y,a,y; MOR y,y,b; ANDN y,x,y`', where $a = $ #`ff7f3f1f0f070301`, $b = $ #`80c0e0f0f8fcfeff`. And to extract leading bits in each byte, we can use '`MOR y,c,x; ANDN y,x,y`', where $c = a \ll 8$.]

Page 589 line 1 of answer 41 ——— 10 Jan 2012

$\sum_{k=1}^{\infty} z^{2^k}/(1-z^{2^k}) \leadsto \sum_{k=1}^{\infty} z^{2^k}/(1-z^{2^k}) = \sum_{k=1}^{\infty} kz^{2^k}/(1-z^{2^{k+1}})$

▶**Page 592** line 8 of answer 58 ——— 02 Feb 2012

$k > 1 \leadsto k > 0$

▶**Page 594** line 1 of answer 71 ——— 28 Jan 2012

$\theta_k \leadsto \theta_k \ll 2^k$

Page 594 lines 4 and 5 of answer 72 ——— 02 Feb 2012

So φ . . . 58(f). $\leadsto$ So $\varphi \in \Omega$ by exercises 5.3.4–11 and 58(f).

▶**Page 594** line 12 of answer 72 ——— 30 Jan 2012

$\theta_1 = (10010010)_2 \leadsto \theta_1 = (10010011)_2$

▶**Page 595** line −6 of answer 75 ——— 31 Jan 2012

$x_7x_5x_5x_3x_1x_1x_0 \leadsto x_7x_5x_5x_3x_3x_1x_1x_0$

Page 597 end of answer 87 ——— 06 Feb 2021

dictionary.] $\leadsto$ dictionary. On `MMIX`, J. Dallos saves one instruction: '`MOR y,m,x; ANDN x,x,y`', where $m = 1 \ll 53$.]

Page 598 new answer 96 ——— 30 Jun 2016

96. If $0 \le x, y < 256$, we have $x < y \iff \bar{x} + y \ge 256 \iff \lfloor(\bar{x}+y)/2\rfloor \ge 128$. Thus we can use (88), replacing x by $\bar{x}$.

To get a similar test function for $[x_j \le y_j] = 1 - [y_j < x_j]$, we just interchange $x \leftrightarrow y$ and take the complement. The result, after noting that $\bar{y} \oplus x = \bar{x} \oplus y$ and $\overline{u+v} = \bar{u} - v$ and $\overline{\bar{y} \,\&\, x} = \bar{x} \mid y$, is analogous to (93):

$$t \leftarrow \big((\bar{x} \mid y) - (((\bar{x} \oplus y) \gg 1) \,\&\, \bar{h})\big) \,\&\, h.$$

Before the final '&h', N. Juffa observes that we have $\lfloor(\bar{x}_j + y_j + 1)/2\rfloor$ in each byte, a bytewise average that is rounded up instead of down when $x_j + y_j$ is odd.

Page 598 new answer 98 ——— 30 Jun 2016

98. Set $t \leftarrow \big((\bar{x} \,\&\, y) + (((\bar{x} \oplus y) \gg 1) \,\&\, \bar{h})\big) \,\&\, h$, $t \leftarrow (t - (t \gg 1)) + t$, $v \leftarrow (x \oplus y) \,\&\, t$, $z \leftarrow x \oplus v$, $w \leftarrow y \oplus v$. [This 13-step procedure uses (93) to find the bytes in which we want to swap $x_j \leftrightarrow y_y$. Of course the `MMIX` solution is much quicker, if available, because $\min(x,y) = x - (x \dot{-} y)$: `BDIF t,x,y; ADDU z,y,t; SUBU w,x,t`.]

Page 598 line 5 of answer 99 ——— 30 Jun 2016

the formulas in answer 96 $\leadsto$ formulas analogous to (91)

▸**Page 602** line 1 of answer 124 ——— 04 Aug 2011

$\{1, 2, \ldots, 2^{n-1}\} \leadsto \{2^0, 2^1, \ldots, 2^{n-1}\}$

Page 605 lines 14 thru −2 of answer 141 ——— 14 Jul 2020

U_n/n appears to converge ... smallest gap $\leadsto$ U_n/n appears to converge to $\rho \approx 13.52$; for example, $U_{10000000} = 135160791$ and $U_{1000000000} = 13517631473$. D. W. Wilson noticed that the numbers form quasi-periodic "clusters" whose centers differ by multiples of $\delta \approx 21.6016$; then S. Steinerberger discovered empirically that $U_n \bmod \lambda$ almost always lies in $[\frac{1}{3}\lambda \,..\, \frac{2}{3}\lambda]$, where $\lambda \approx 2.443443$ [Report DCS/TR-1508 (Yale University, 2015)]. By exploiting that amazing property, P. Gibbs [viXra:1508.0085 (2015)] found that roughly $O(N)$ time and $O(N)$ space suffice to compute the first N Ulam numbers. He and Jud McCranie calculated more than 25 billion Ulam numbers in 2015, discovering large gaps such as $U_{23647775834} - U_{23647775833} = 319654122989 - 319654121875 = 1114$. The smallest gap

Page 607 last line of answer 153 ——— 28 Jan 2022

262.] $\leadsto$ 262; see also O. Darwish, A. Elmasry, and J. Katajainen, *ACM Transactions on Algorithms* **17** (2021), 18:1–18:19.]

▸**Page 608** line 3 ——— 25 Feb 2012

six operations $\leadsto$ seven operations

Page 608 in answer 159 ——— 24 Apr 2012

line 5: 20-step $\leadsto$ 19-step
line 9: $z \leftarrow z \oplus (t \,\&\, (z \oplus ((w{+}1) \gg 1)))$ $\leadsto$ $z \leftarrow (z \mid t) - (t \gg 1)$

▸**Page 608** line −3 of answer 159 ——— 03 Mar 2012

$t \leftarrow y \,\&\, (y-1) \,\&\, \bar{\mu}_0$, $y \leftarrow y - t + [t \ne 0]((t-1) \,\&\, \mu_0)$ $\leadsto$
$t \leftarrow y \,\&\, {-y} \,\&\, \bar{\mu}_0$, $y \leftarrow (t{=}0?\ y : y - 1 - ((t{-}1) \,\&\, \bar{\mu}_0))$

Page 608 line 2 of answer 161 ——— 03 Mar 2012

strips can $\leadsto$ strips, $y \bmod 2$, can

▶**Page 610** replacement for step C1 in answer 168 ——— 07 Mar 2012

C1. [Loop on k.] Set $A_{j0} \leftarrow 0$ for $0 \le j < M$. Then do step C2 for $k = 1, 2, \ldots, N'$. Then go to C5.

Page 610 lines −4 and −3 of answer 168 ——— 07 Mar 2012

An $M \times N$ ... even (161). $\rightsquigarrow$ In many cases, like (157) and (159) and even (161), an $M \times N$ torus is equivalent to an $(M-1) \times (N-1)$ array surrounded by zeros.

Page 611 end of answer 173 ——— 16 May 2015

15–36.] $\rightsquigarrow$ 15–36. See also J. Serra, *Image Analysis and Mathematical Morphology* (1982).]

▶**Page 611** replacement for line 3 of answer 176 ——— 08 Mar 2012

(b) Some $w' \in G'$ is adjacent or equal to each vertex of $N_u \cup N_v$. (Four cases.)

Page 612 lines 12–14 ——— 23 Jun 2018

If $\mathtt{P} = \mathtt{P}'$ and ... $\mathtt{Q}' \leftarrow$ `PARENT(Q)`. $\rightsquigarrow$ If $\mathtt{P} = \mathtt{P}'$, then $\mathtt{Q} \leftarrow \mathtt{Q}' \leftarrow$ `CHILD(P)` if `CHILD(P)` $\neq \Lambda$, otherwise $\mathtt{Q} \leftarrow \mathtt{P}$ and $\mathtt{Q}' \leftarrow$ `PARENT(Q)`. If $\mathtt{P} \neq \mathtt{P}'$, then $\mathtt{Q} \leftarrow \mathtt{Q}' \leftarrow$ `SIB(P)` if `SIB(P)` $\neq \Lambda$, otherwise $\mathtt{Q} \leftarrow$ `PARENT(P)` and $\mathtt{Q}' \leftarrow$ `PARENT(Q)`.

Page 612 line 4 of step I4 ——— 11 Mar 2012

$\mathtt{R}' \leftarrow \mathtt{R}$, $\mathtt{R} \leftarrow$ `SIB(R)` $\rightsquigarrow$ $\mathtt{R} \leftarrow$ `SIB(R)`

▶**Page 613** in answer 181 ——— 16 Mar 2012

$b\eta \rightsquigarrow b(\eta + \frac{1}{2})$ [twice] and $b\xi \rightsquigarrow b(\xi + \frac{1}{2})$ [twice]

▶**Page 614** line 6 of answer 185 ——— 21 Mar 2012

see exercise 183 $\rightsquigarrow$ see exercise 184

▶**Page 615** lines 1 and 2 ——— 23 Mar 2012

$> 0 \rightsquigarrow \ge 0$ [twice]

▶**Page 617** lines 21–22 of answer 192 ——— 27 Mar 2012

So $F_{2(2^{2e}-1)}(y) = yF_{2^{2e}-1}(y)^2$ is a multiple of $(x^{2q}+1)/(x^2+1)$ $\rightsquigarrow$ So $F_{4(2^{2e}-1)}(y) = y^3F_{2^{2e}-1}(y)^4$ is a multiple of $(x^{2q}+1)/(x^2+1) = f_2(x)^2 f_3(x)^2 \ldots f_r(x)^2$

▶**Page 618** line 5 ——— 28 Mar 2012

$\operatorname{lcm}(c_1, \ldots, c_r) < n \rightsquigarrow \operatorname{lcm}(c_1, \ldots, c_r) \le n$

Page 619 line 1 of answer 203 ——— 13 Feb 2018

`b` = `#0f0f0f0f0f0f0f0f` $\rightsquigarrow$ `b` = `#0804020108040201`

Page 620 replacement for answer 214 ——— 08 Aug 2020

214. (Solution by Adam P. Goucher.) By considering the irreducible factors of X's characteristic polynomial, we must have $X^{n-1} = I$ or $X^n = I$, where $n = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 17 \cdot 31 = 1328040$, since $n \bmod 127 = 1$. Hence the following sequence of 28 instructions, based on an addition chain of length $l(n-2) = 25$, will compute $Y = X^{-1}$: `MXOR $1,x,x;` `MXOR $2,$1,x;` `MXOR y,$2,$1;` S^3; `MXOR $1,y,x;` `MXOR y,$1,y;` S^6; `MXOR y,y,$2;` S^6; `MXOR y,y,$1;` S; `MXOR y,y,x;` S; `MXOR $1,y,x;` `CMPU $2,$1,i;` `CSNZ y,$2,$1;` here S stands for '`MXOR y,y,y`' and `i` = `#8040201008040201` is the identity matrix. To test if X is nonsingular, do `MXOR $1,y,x` and compare `$1` to `i`.

Page 620 improvement to answer 218 ——— 13 Apr 2012

line 4: for $1 < k < d$: Set $s \leftarrow 0$, $j \leftarrow 1$; then while $x \neq 1$ $\leadsto$
for $1 < k < d$, assuming that $t_k = 2^k$ for $k \geq d/2$: Set $s \leftarrow 0$, $j \leftarrow 1$; then while $j < d/2 - 1$

also append a new sentence to the end of the first paragraph:
Finally set $s \leftarrow (s + 1 - x) \bmod 2^d$.

also append a new sentence to the end of the second paragraph:
(With another multiplication we could return $(1 - s)a$ as soon as $j \geq d/2$.)

Page 621 replacement for first four lines ——— 04 Aug 2011

Suitable numbers t_k can be computed by setting $t_k \leftarrow 1 \ll k$ for $d - 1 \geq k \geq d/2$ and proceeding as follows for the remaining ks, in decreasing order: Set $x \leftarrow 1+(1\ll k)$, $x \leftarrow x+(x\ll k)$, $s \leftarrow 0$, $j \leftarrow k$; then while $j < d/2-1$ set $j \leftarrow j+1$, and if $x \,\&\, (1\ll j) \neq 0$ also set $x \leftarrow x + (x \ll j)$, $s \leftarrow s - t_j$; finally $t_k \leftarrow (s + x - 1) \gg 1$. For example, when $d = 32$ we get $t_{15} =$

Page 621 new answer ——— 16 Nov 2013

219. It "sorts" the bits, changing x to $2^{\nu x} - 1$.

Page 621 answer 4 ——— 30 Jun 2016

(See 7.1.3–(93).) $\leadsto$

Page 623 replacement for answer 15 ——— 23 Apr 2021

15. Both cases lead by induction to well known sequences of numbers: (a) The Lucas numbers $L_n = F_{n+1} + F_{n-1} = \phi^n + \hat{\phi}^n$ [see E. Lucas, *Théorie des Nombres* (1891), Chapter 18]. (b) The Perrin numbers, defined by $P_3 = 3$, $P_4 = 2$, $P_5 = 5$, $P_n = P_{n-2}+P_{n-3} = \rho^n+\hat{\rho}^n+\overline{\hat{\rho}}^n$. [See E. Lucas, *Association Française pour l'Avancement des Sciences*, Compte-rendu **5** (1876), 62; R. Perrin, L'*Intermédiaire des Mathématiciens* **6** (1899), 76–77; Z. Füredi, *Journal of Graph Theory* **11** (1987), 463.]

Page 632 append new sentence to line 5 of answer 77 ——— 23 Oct 2012

We have $(c_0, c_1, c_2, \ldots) =$
$(1, 1, 2, 1, 5, 3, 5, 1, 19, 14, 25, 6, 50, 14, 19, 1, 167, 148, 282, 84, \ldots)$.

▸**Page 632** end of second paragraph in answer 77 ——— 25 Aug 2012

$y_{2^n+k} = 0 \leadsto y_{2^{n-1}+k} = 0$

Page 641 line −3 of answer 125 ——— 23 Apr 2021

the plastic constant χ $\leadsto$ the plastic constant ρ

Page 643 line 3 of answer 133 ——— 07 Nov 2012

$j = 2^{x-1}$ and $i \,\&\, j \neq 0\}$ $\leadsto$ $1 \leq x \leq n$, $j = 2^{x-1}$, and $i \,\&\, j \neq 0\}$

Page 646 line −2 of answer 138 ——— 27 Nov 2012

method for $m = 1$ $\leadsto$ method

Page 653 lines 2 and 3 of answer 169 ——— 22 Dec 2017

unexplained. A paper ... "curious research group."] $\leadsto$ unexplained.]

Page 653 lines 5 and 6 of answer 171 ——— 10 Mar 2013

where $\circ$ is a nontrivial binary operator and we have $\leadsto$ where g and h are read-once and regular and $\circ$ is a nontrivial binary operator, and where we have

▸**Page 657** line 2 of answer 183 ———— 29 Nov 2013

$k > 0 \leadsto k > 1$

Page 664 last line of answer 215 ———— 03 Mar 2021

#R126 becomes #R126, 1–20

▸**Page 665** line 5 ———— 30 Dec 2013

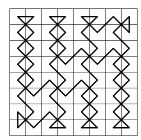
and with neglible $\leadsto$ and uses negligible

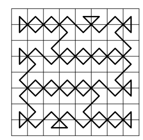
Page 665 line 1 of answer 218 ———— 31 May 2017

the matrix $\leadsto$ the incidence matrix

Page 668 replacement for second paragraph of answer 232 ———— 28 Sep 2017

Essentially only two king's tours, shown here, achieve the maximal length $8 + 56\sqrt{2}$. [Nikolai Beluhov has proved that the longest $n \times n$ king's tours, when n is even, have length $n + n(n-1)\sqrt{2}$. See the Bulgarian journal *Matematika Plyus* **14**, 4 (October–December 2006), 61–64.]





Page 670 line 5 ———— 30 Jan 2013

as in (52) $\leadsto$ as explained in answer 198

▸**Page 670** line 5 ———— 21 2011

If $v < g_o$, $\leadsto$ If $f_o > g_o$, cache and return $\mathrm{AND}(\mathrm{NONSUB}(f, g_l), \mathrm{NONSUB}(f, g_h)$. Otherwise set $v \leftarrow f_o$. If $v < g_o$,

Page 670 line 11 ———— 18 Jan 2022

$v = f_o \leadsto v \leftarrow f_o$.

Page 670 line 16 ———— 15 May 2015

$r_h \leftarrow \mathrm{NONSUP}(r, r_l) \leadsto r_h \leftarrow \mathrm{BUTNOT}(r, r_l)$

▸**Page 670** line 20 ———— 08 Dec 2010

by $\epsilon \in g$' $\leadsto$ by '$\epsilon \in g$'

Page 672 line 3 ———— 23 May 2019

8-queens problem $\leadsto$ 8 queens problem

Page 672 line 9 of answer 241(c) ———— 15 Feb 2016

Tit Bits (1 Jan 1898) $\leadsto$ *Tit-Bits* **33** (1 January 1898)

Page 673 line −4 of answer 241 ———— 03 Mar 2021

8 (2001), #R29; $\leadsto$ **8**, 1 (2001), #R29, 1–19 and corrigenda;

▸**Page 673** line 1 of answer 242 ———— 23 May 2019

3-regular hypergraph $\leadsto$ 3-uniform hypergraph

Page 673 line −4 of answer 242 ———— 23 May 2019

of an 8-point kernel $\leadsto$ of 8-point kernels

▸**Page 677** line −2 of answer 262 ———— 30 Jul 2017

$C(a) \leadsto C(f)$

Page 679 line 1 of answer 8 ——— 23 Jun 2020

parity bit ... right $\rightsquigarrow$ parity bit a_{-1} at the right

Page 679 in answer 10 ——— 19 Aug 2021

line 4: *Shūki Sanpō* (1769), Problem 44, $\rightsquigarrow$ *Shūki Sanpō* **2** (1769), 20–21,
line 5: puzzle after a certain number of steps. $\rightsquigarrow$ puzzle.

Page 679 new penultimate paragraph for answer 10 ——— 25 Jun 2016

A. M. Hinz has discovered that the German physicist G. C. Lichtenberg published similar results in *Göttingische Anzeigen von gemeinnützigen Sachen* **1** (1769), 637–640.

Page 679 replacement for final paragraph of answer 10 ——— 19 Aug 2021

L. Pacioli explained how to solve the ring puzzle in Chapter 107 of *De Viribus Quantitatis* (unpublished, c. 1500). Then G. Cardano mentioned it in Book 15 of his *De Subtilitate Libri XXI* (Nuremberg: 1550), calling it "useless but admirably subtle." With the trick of exercise 11, he explained how to make a 95-move tour of every relevant configuration of seven rings. J. Wallis devoted seven pages to these "complicati annuli" in the Latin edition of his *Algebra* **2** (Oxford: 1693), Chapter 111, presenting detailed methods for the nine-ring case. He included the operation of sliding a ring through the bar as well as putting it on or off, but hinted that shortcuts were available. [See A. Heeffer and A. M. Hinz, *Recreational Math. Magazine* **4**, 8 (December 2017), 5–23.]

Page 679 replacement for Algorithm C in answer 12 ——— 12 May 2016

C1. [Initialize.] Set $t \leftarrow 1$, $s_1 \leftarrow n$. (We assume that $n > 1$.)

C2. [Visit.] Visit $s_1 \ldots s_t$. Then go to C4 if t is even.

C3. [Odd step.] If $s_t > 1$, set $s_t \leftarrow s_t - 1$, $s_{t+1} \leftarrow 1$, $t \leftarrow t + 1$; otherwise set $t \leftarrow t - 1$ and $s_t \leftarrow s_t + 1$. Return to C2.

C4. [Even step.] If $s_{t-1} > 1$, set $s_{t-1} \leftarrow s_{t-1} - 1$, $s_{t+1} \leftarrow s_t$, $s_t \leftarrow 1$, $t \leftarrow t + 1$; otherwise set $t \leftarrow t - 1$, $s_t \leftarrow s_{t+1}$, $s_{t-1} \leftarrow s_{t-1} + 1$ (but terminate if t has become 1). Return to C2. ▮

Page 680 replacement for lines 4–7 ——— 12 May 2016

B3. [Odd step.] If $q_{t-1} > 1$, set $q_t \leftarrow 1$ and $t \leftarrow t + 1$; otherwise set $t \leftarrow t - 1$. Return to step B2.

B4. [Even step.] If $q_{t-2} > q_{t-1} + 1$, set $q_t \leftarrow q_{t-1}$, $q_{t-1} \leftarrow q_t + 1$, $t \leftarrow t + 1$; otherwise set $q_{t-2} \leftarrow q_{t-1}$ and $t \leftarrow t - 1$ (but terminate if t has become 1). Return to B2. ▮

Page 680 notation change in answer 13 ——— 12 May 2016

$j \rightsquigarrow t$ (in six subscripts)

Page 684 replacement for lines 3–7 of answer 33 ——— 25 Jan 2014

for $0 \le x < \frac{1}{2}$, because $r_1(x) = 1$ in this range and $r_j(2x) = r_{j+1}(x)$ for all x. On the other hand when $\frac{1}{2} \le x < 1$,

$$\begin{aligned} w_k(x) = (-1)^{\lceil k/2 \rceil} w_{\lfloor k/2 \rfloor}(2x) &= (-1)^{\lceil k/2 \rceil} r_1(2x)^{b_1+b_2} r_2(2x)^{b_2+b_3} \ldots \\ &= r_1(x)^{b_0+b_1} r_2(x)^{b_1+b_2} r_3(x)^{b_2+b_3} \ldots \end{aligned}$$

because $\lceil k/2 \rceil \equiv b_0 + b_1$ (modulo 2) and $r_1(x) = -1$ in this range.

Page 684 append two new lines to answer 33 ———— 22 Apr 2015

(Incidentally, the Rademacher functions had been discovered independently by Felix Hausdorff in 1915 but not published; see his *Gesammelte Werke* **5** (2006), 757–758.)

Page 685 line 3 of answer 39(b) ———— 12 Mar 2021

Philos. Mag. **34** $\rightsquigarrow$ *Philos. Mag.* (4) **34**

Page 685 last line of answer 39 ———— 12 Mar 2021

roots of unity.) $\rightsquigarrow$ roots of unity. See also the prior work of T. P. Kirkman, *Philosophical Mag.* (3) **33** (1848), 447–459, 494–509; **35** (1850), 292–301.)

Page 686 lines 2–5 of answer 43 ———— 15 Sep 2014

Exact calculation ... in 1972.) $\rightsquigarrow$ And indeed, H. Haanpää and P. R. J. Östergård found the exact value $d(6) = 71{,}676{,}427{,}445{,}141{,}767{,}741{,}440$ in 2011, by using symmetry and "gluing together" disjoint paths whose endpoints x have $\nu x = 3$ and whose interior vertices have $\nu x \leq 3$. [*Math. Comp.* **83** (2014), 979–995.]

▸**Page 690** line 1 of answer 68 ———— 27 May 2021

take $(n-1)$-trit modular $\rightsquigarrow$ take the $(n-1)$-trit modular

▸**Page 691** replacement for step X4 in answer 73 ———— 20 Jul 2018

X4. [Compute π_n.] Set $p'[0] \leftarrow n-1$, and $p'[j] \leftarrow p[p[j-1]]$ for $1 \leq j < n$.

▸**Page 692** line 4 of answer 74 ———— 27 May 2013

$2\bigl(\binom{n-1}{j-1} + \binom{n-1}{j} + \binom{n-1}{j+1}\bigr) - 1 \rightsquigarrow \leq 2\bigl(\binom{n-1}{j-1} + \binom{n-1}{j} + \binom{n-1}{j+1}\bigr) - 1$

Page 692 last line of answer 74 ———— 15 Jul 2013

7.10–00 $\rightsquigarrow$ 7.5.6–00

▸**Page 694** line 1 of answer 84 (in the eBook only) ———— 15 Jan 2015

solution in $\rightsquigarrow$ solution in

Page 696 line -4 of answer 91 ———— 03 Mar 2021

3 (1996) $\rightsquigarrow$ **3**, 1 (1996)

▸**Page 696** replacement for last three lines of answer 91 ———— 02 Apr 2014

Incidentally, it is amusing to note that the mapping $k \mapsto g(2k)$ is a one-to-one correspondence between all binary n-tuples with no consecutive 1s and all binary $(n+1)$-tuples with no odd-length runs.

▸**Page 697** lines 7 and 8 of answer 95 ———— 24 Apr 2014

the cycle of $f()$ ... By (58) $\rightsquigarrow$ the cycle of $f()$ and $f'()$ with x^n changed to x^{n+1} and the b's are the same cycle with x^n changed to x^{n-1}, for $0 \leq x < r$. Hence we have $b_k \ldots b_{k+n-1} = a_{k+\delta_k} \ldots a_{k+n-1+\delta_k}$ for all k, where δ_k is even. By (58)

▸**Page 699** in answer 99 ———— 21 Jul 2013

line 6: $j^+ \equiv k^+ \rightsquigarrow j^+ \equiv k^-$
lines 7–9: we may take ... Then $\rightsquigarrow$ we may take $l/2 = k^- + (2^n-2)(2^{n-3}(j^+ - k^-) \bmod (2^{n-1}+1))$ if $j^+ \geq k^-$, otherwise $l/2 = j^+ + (2^n+2)(2^{n-3}(k^- - j^+) \bmod (2^{n-1}-1))$. Otherwise let $(l+1)/2 = k^+ + (2^n+2)x = j^- + 1 + (2^n-2)y$. Then

▶**Page 700** in answer 105(b) ——— 22 Dec 2013

is prime. The condition ... would begin $\leadsto$ is prime and $|\lambda_1| = r$. If $a_j \neq a_{j+r}$ for some j, we must have $a_j < a_{j+r}$ for the smallest such j, because $\lambda_1\beta\omega < \beta\omega$. But then α would begin

Page 701 in answer 108 ——— 12 Jun 2014

line 1: $\lambda_{k+1} \leadsto \beta\alpha < \lambda_{k+1}$
line 8: suffix of λ_{k-2}, $\leadsto$ suffix of λ_{k-2} because $|\beta'| \leq n/2$,

Page 701 append to the last line of answer 108 ——— 12 May 2016

[See E. Moreno and D. Perrin, *Advances in Applied Mathematics* **33** (2004), 413–415; **62** (2015), 184–187.]

Page 701 append new paragraph to answer 109 ——— 23 Jul 2020

[Binary examples were first considered by I. S. Reed and R. M. Stewart, *IRE Trans.* **IT-8** (1962), 10–12, who called them "perfect maps"; they allowed also certain generalized toruses as in exercise 7–137. Many subsequent authors decided to support only the $2^{mn}-1$ *nonzero* settings of $m \times n$ windows; see F. J. MacWilliams and N. J. A. Sloane, *Proc. IEEE* **64** (1976), 1715–1729.]

▶**Page 701** line 2 of answer 110 ——— 15 Jun 2014

$a_{j+(i-1)/2 \bmod m^2} \leadsto a_{j+(i+1)/2 \bmod m^2}$

▶**Page 705** in answer 17 ——— 02 Aug 2014

line 7: $c_0 = -1 \leadsto c_0 \leftarrow -1$
line -2: $i = j + o_k \leadsto i \leftarrow j + o_k$

Page 706 last line of answer 20 ——— 20 Dec 2018

octahedral group $\leadsto$ hyperoctahedral group $\mathcal{B}_n$

▶**Page 707** line 3 of answer 30 ——— 06 May 2019

Math $\leadsto$ *Math.*

Page 711 last line of answer 55 ——— 09 Nov 2014

$\beta_{n!}^{-j}\alpha_m$ fixes n and $\leadsto$ α_m fixes n, $\beta_{n!}$ fixes n, and

Page 714 replacement for answer 71 ——— 31 Jul 2020

71. Yes — including one that has a reasonably simple succession rule. See J. Sawada and A. Williams, *ACM Transactions on Algorithms* **16** (2020), #11.1–#11.17.

Page 715 first line of answer 77 ——— 03 Jan 2020

`\'a la Heap` $\leadsto$ `à la Heap`

Page 717 line 2 ——— 28 Apr 2014

[say '⟨Visit the permutation in register `a`⟩', thus matching the typographic style of (45)]

Page 718 improvement to line 3 ——— 25 Jan 2015

$k'_{j+1} \geq k_j$. $\leadsto$ $k'_{j+1} \geq k_j$, because $[k_j \,..\, k_{j+1}]$ is not a chain.

Page 719 improvement to answer 95 — 28 Jan 2015

Generate ...! Algorithm V. $\rightsquigarrow$ Using Algorithm V with the relations $1 \prec n \succ 2 \prec n-1 \succ \cdots$, visit the up-down permutations $a'_1 a'_n a'_2 a'_{n-1} \ldots$.

▶**Page 719** line 1 of answer 100 — 01 Feb 2015

$l[q] \rightsquigarrow l_q$

Page 720 replacement for top three lines — 18 Feb 2015

Y3. [Decrease k.] Set $k \leftarrow k-1$, and terminate if $k = 0$. Otherwise set $q \leftarrow u_k$, $p \leftarrow a_q$, and $r \leftarrow l_q$. If $p = q$, set $q \leftarrow 0$ and $k \leftarrow k+1$ (preparing to make $a_p > p$). Otherwise set $l_{u_{k-1}} \leftarrow q$ (preparing to make $a_p > q$).

Page 720 line 11 — 18 Feb 2015

$11a_n + 12t_n$, where $a_n < 1.25331414t_n$. $\rightsquigarrow$ $9a_n + 12t_n$. It can be shown that $\sum a_n z^n/n! = e^z(z/1 + z^3/(1\cdot 3) + z^5/(1\cdot 3\cdot 5) + \cdots)$, and that $a_n \sim \sqrt{\pi/2}\,t_n$.

▶**Page 720** last line of answer 104 — 23 Aug 2020

0, 0, 0, 2, 6 $\rightsquigarrow$ 1, 0, 0, 2, 6

▶**Page 720** new last line for answer 104 — 23 Aug 2020

[See E. I. Marshall, *Sankhyā* **B56** (1994), 59–66.]

▶**Page 720** line 3 of answer 105 — 10 Aug 2018

weak orders. $\rightsquigarrow$ weak orders.)

Page 722 line 8 of answer 111 — 17 Aug 2016

[G. Hurlbert $\rightsquigarrow$ [*Notes:* C. Bachet presented a similar idea for $n = 4$ already in his *Problemes plaisans et delectables* (Lyon: 1612), 123, but with two half-cycles instead of a full cycle. G. Hurlbert

▶**Page 728** lines 1–3 of answer 23 — 05 Aug 2015

(If $r = 1$ we have ... etc.) $\rightsquigarrow$ (If $t-r$ is even we have $j > r \iff c_r = r-1$; otherwise $j > r \iff c_{r+1} = c_r + 1$ and $c_{r-1} = r-2$, assuming that $c_0 = -1$.)

Page 734 in answer 53 — 30 Jan 2013

46. $\rightsquigarrow$ 46, for solutions to some small cases.

Page 734 another sentence for answer 53 — 14 Nov 2016

T. Verhoeff [*Designs, Codes and Cryptography* **84** (2017), 295–310] has found important structural information, which yields a general solution for $d = 1$ and probably for all d.

Page 735 new answer 56 — 16 Sep 2021

56. [*Discrete Mathematics* **48** (1984), 163–171.] Such a solution resolves the "middle levels conjecture" [I. Havel, *Teubner-Texte zur Mathematik* **59** (1983), 101–108], which states that there is a Gray path through all binary strings of length $2t-1$ and weights $\{t-1, t\}$. In fact, such strings can be generated by a delta sequence of the special form $\alpha_0\alpha_1\ldots\alpha_{2t-2}$, where the elements of α_k are those of α_0 shifted by k, modulo $2t-1$. For example, when $t = 3$ we can start with $a_5a_4a_3a_2a_1a_0 = 000111$ and repeatedly swap $a_0 \leftrightarrow a_\delta$, where δ runs through the cycle (4134 5245 1351 2412 3523). [See T. Mütze, *Proc. London Math. Soc.* (3) **112** (2016), 677–713; T. Mütze and J. Nummenpalo, *ACM Transactions on Algorithms* **14** (2018), 15:1–15:29; A. Merino, O. Mička, and T. Mütze, *SODA* **32** (2021), 563–617.]

▶**Page 746** line 3 of answer 109 ——— 27 Jul 2018

and(000111202033332221313) $\rightsquigarrow$ and (000111202033332221313)

▶**Page 751** line −2 of answer 25 ——— 22 Mar 2021

$B_2B_1t \rightsquigarrow -B_2B_1t/2$

▶**Page 751** line 4 of answer 27 ——— 23 Jul 2014

(29) $\rightsquigarrow$ (29)

Page 755 a better solution to answer 54(b) ——— 26 Jul 2013

(b) True, but not so obvious. The Ferrers diagram shows us that

$$a'_1 + \cdots + a'_k \;=\; \textstyle\sum_{j=1}^{\infty} \min(k, a_j);$$

thus we want to show that $\alpha \succeq \beta$ implies $\sum_{j=1}^{\infty} \min(k, a_j) \le \sum_{j=1}^{\infty} \min(k, b_j)$ for all $k \ge 0$. This inequality is clear when $k \ge b_1$; and if $b_l \ge k \ge b_{l+1}$ we have $\sum_{j=1}^{\infty} \min(k, a_j) \le \sum_{j=1}^{l} k + \sum_{j=l+1}^{\infty} a_j \le kl + \sum_{j=l+1}^{\infty} b_j = \sum_{j=1}^{\infty} \min(k, b_j)$.

Page 759 replacement for last three lines of answer 64 ——— 12 Aug 2019

This construction is due to T. Colthurst and M. Kleber, arXiv:0907.3873 [math.CO] (2003), 6 pages. Euler had considered the number of such partitions in §50 of his paper in 1750.]

Page 761 end of answer 73 ——— 17 May 2015

476.] $\rightsquigarrow$ 476. This is a special case of considerably more general results proved by N. J. Fine in 1959; see R. A. Gilbert, *AMM* **122** (2015), 322–331.]

Page 765 append to answer 19(a) ——— 01 Oct 2015

[See J. Kürschák, *Matematikai versenytételek* (Szeged: 1929), Problem 2 from the Eötvös Mathematical Competition of 1928.]

Page 768 last line of answer 27 ——— 03 Mar 2021

paper #R15 $\rightsquigarrow$ #R15, 1–32

▶**Page 770** line 7 of answer 34 ——— 25 Feb 2011

at most three times $\rightsquigarrow$ at most four times

▶**Page 770** line 10 of answer 34 ——— 25 Feb 2011

all used the $\rightsquigarrow$ all used coalescences of the

Page 772 line 4 of answer 43 ——— 29 Jan 2018

$|t| > \pi \rightsquigarrow |t| \ge \pi$

▶**Page 775** line −4 of answer 53 ——— 13 Sep 2011

[See $\rightsquigarrow$ See

Page 777 last line of answer 62 ——— 05 Sep 2012

A1, 1–13 $\rightsquigarrow$ A1:1–A1:13; **5** (2005), A9:1

▶**Page 777** last line of answer 63 ——— 22 May 2019

297–303 $\rightsquigarrow$ 279–303

Page 778 replacement for answer 66 ——— 05 Jan 2012

66. (The maximum of (48) when $n = 100$ is achieved only for the three partitions $7^1 6^2 5^4 4^6 3^7 2^6 1^4$, $7^1 6^2 5^4 4^6 3^8 2^5 1^3$, $7^1 6^2 5^4 4^7 3^6 2^6 1^3$.)

▶**Page 778** line 5 of answer 72 — eBook version only ——— 05 Feb 2017

See Table ?? $\rightsquigarrow$ See Table A–1.

Page 780 last lines of answer 82 ——— 17 Sep 2016

[This problem ... (1972).] or [This exercise ... (1972).] $\rightsquigarrow$ [The special cases with common sums 5/2 and 10 had been noticed earlier by H. E. Dudeney in *Strand* **68** (1924), 422, 530.]

▶**Page 783** line −3 of answer 18 ——— 12 Aug 2020

answer 19 $\rightsquigarrow$ answer 17

Page 785 last two lines of answer 22 ——— 08 Aug 2011

351–357] ... Algorithm B.) $\rightsquigarrow$ 351–357; **54** (2011), 776–785] has shown that this algorithm, Algorithm P, and Algorithm B can all be extended to t-ary trees in interesting ways.)

▶**Page 788** replacement for answer 30 ——— 13 Aug 2020

30. (a) $\bar f_{n-1} \ldots \bar f_1$, because internal node j in symmetric order has a nonempty right subtree if and only if internal node $j+1$ in symmetric order has an empty left subtree.

(b) By answer 26(f), the general canopy $1^{p_1} 0^{q_1+1} 1^{p_2+1} 0^{q_2+1} \ldots 1^{p_k+1} 0^{q_k}$ arises from binary trees whose nodes in symmetric order have the specification $R^{p_1} N L^{q_1} B R^{p_2} N L^{q_2} B \ldots R^{p_k} N L^{q_k}$, where B means "both subtrees are nonempty," R means "the right subtree is nonempty but not the left," L means "the left subtree is nonempty but not the right," and N means "neither subtree is nonempty."

The number $f(p_1, q_1; \ldots; p_k, q_k)$ of such trees is $\binom{p_1+q_1}{p_1}$ when $k = 1$; otherwise it's

$$[p_1 > 0]\, f(p_1 - 1, q_1; \ldots; p_k, q_k) + \sum_{j=1}^{k-1} f(p_1, q_1; \ldots; p_j, q_j)\, f(p_{j+1}, q_{j+1}; \ldots; p_k, q_k) + [q_k > 0]\, f(p_1, q_1; \ldots; p_k, q_k - 1).$$

And in particular, $f(1,0;0,0;1,0;5,3;0,0;0,0;0,2;0,0;1,2) = 114044694$.

(c) $d_j = 0$ if and only if $c_{j+1} > c_j$, by exercise 3.

(d) In general, the canopy of $F \perp F'$ is $f_1 \ldots f_{n-1} \wedge f'_1 \ldots f'_{n-1}$, by exercise 27(a); the canopy of $F \top F'$ is $f_1 \ldots f_{n-1} \vee f'_1 \ldots f'_{n-1}$, by (a) and exercise 27(d).

[The fact that complements always exist in the Tamari lattice is due to H. Lakser; see G. Grätzer, *General Lattice Theory* (1978), exercise I.6.30. X. G. Viennot coined the nice term "canopy"; see arXiv:0905.3081 [math.CO] (2009), 12 pages.]

Page 788 replacement for answer 31 ——— 13 Aug 2020

31. (a) 2^{n-1}; see exercise 6.2.2–5.

(b) $c_1 \le \cdots \le c_n$; $d_1, \ldots, d_{n-1} \le 1$; $e_j > 0$ implies $e_j + \cdots + e_n = n - j$; $k_{j+1} \le k_j + 1$; $p_1 \le \cdots \le p_j \ge \cdots \ge p_n$ for some j; $s_j > 0$ implies $s_j = n - j$; $u_1 \ge \cdots \ge u_n$; $z_{j+1} \le z_j + 2$. (Other constraints, which apply in general, whittle down the number of possibilities to 2^{n-1} in each case. For example, $u_1 \ldots u_n$ must be a valid sequence of scope coordinates.)

(c) True in only n cases out of 2^{n-1}. (But F^T *is* degenerate.)

(d) The degenerate forest with canopy $f_1 \ldots f_{n-1}$ has $c_{j+1} = c_j + f_j$. Elements $j < k$ are siblings if and only if $f_j = f_{j+1} = \cdots = f_{k-1} = 0$. Thus if F'' is the degenerate forest with canopy $f_1 \ldots f_{n-1} \wedge f'_1 \ldots f'_{n-1}$, then $F'' \mathrel{|\!<} F$ and $F'' \mathrel{|\!<} F'$; hence $F'' \mathrel{|\!<} F \barwedge F' \dashv F \perp F'$. And we also have $F \perp F' \dashv F''$ by (b). A similar argument proves that $F \veebar F' = F \top F'$ is the degenerate forest with canopy $f_1 \ldots f_{n-1} \vee f'_1 \ldots f'_{n-1}$.

Thus, when the Kreweras and Tamari lattices are restricted to degenerate forests, they become identical to the Boolean lattice of subsets of $\{1, \ldots, n-1\}$. [This result, in the case of Tamari lattices, is due to George Markowsky, *Order* **9** (1992), 265–290, whose paper also shows that Tamari lattices enjoy many further properties.]

Page 790 new paragraph appended to answer 36 ———— 09 Jun 2012

[The "Fuss–Catalan numbers" $C_n^{(t)}$ were first studied by N. Fuss, *Nova acta acad. scient. imp. Pet.* **9** (1791), 243–251.]

Page 799 in answer 71 ———— 09 Dec 2015

line 1 and line 5 and line 7: $M_{tn} \rightsquigarrow M_{t,n}$
line 6: $M_{(2)n} \rightsquigarrow M_{2,n}$

Page 799 line 1 of answer 72 ———— 09 Dec 2015

$M_{sn} \rightsquigarrow M_{s,n}$

Page 799 in answer 74 ———— 09 Dec 2015

$M_{(p+1)n} \rightsquigarrow M_{p+1,n}$
[Also $M_{0(29)} \rightsquigarrow M_{0,29}$, etc.; insert commas and remove parentheses in the double-subscripts of M.]

Page 799 and 800 in answer 75 ———— 09 Dec 2015

[Insert commas and remove parentheses as in answer 74.]

Page 801 replacement for lines 1–10 of answer 85 ———— 09 Dec 2015

85. (Solution by Günter Rote.) The vector space V can be represented as

$$\{x \mid Ax = 0\}$$

for some $m \times n$ matrix A. Let v_j be the jth column of A, and notice that v_j cannot be zero because V contains no unit vectors. We can therefore assume that $\min(\|v_1\|, \ldots, \|v_n\|) = 1$, after multiplying A by an appropriate constant. Exercise 84 now applies.

Page 802 line −2 of answer 90 ———— 01 Sep 2020

(Two $\rightsquigarrow$ If $p''_j = 0$, $p_{j+m+1} = 0'$ is the second centroid. (Two

Page 802 end of answer 91 ———— 12 May 2021

25.] $\rightsquigarrow$ 25. For random free trees, see H. S. Wilf, *Journal of Algorithms* **2** (1981), 204–207.]

Page 812 new sentence at end of answer 8 ———— 09 Oct 2013

G. W. Leibniz had in fact discovered this rule in an unpublished manuscript dated 12 January 1684; see E. Knobloch, *Archive for Hist. Exact Sciences* **12** (1974), 142–173.

▸**Page 813** last line of answer 15 ———— 16 Sep 2018

$56 + 35 + 20 + 10 + 4 + 1 = 252 \rightsquigarrow 126 + 70 + 35 + 15 + 5 + 1 = 252$

Page 816 first line of answer 32 ——— 24 Oct 2019

$20\cdot 1 = 7 + 13\cdot 1 = 2\cdot 7 + 6\cdot 1 = 10 + 10\cdot 1 = 10 + 7 + 3\cdot 1 = 2\cdot 10.$ ⇝
$20\cdot 1 = 1\cdot 7 + 13\cdot 1 = 2\cdot 7 + 6\cdot 1 = 1\cdot 10 + 10\cdot 1 = 1\cdot 10 + 1\cdot 7 + 3\cdot 1 = 2\cdot 10.$

Page 817 (not an error!) ——— 25 May 2011

[The fact that no text appears above the authors' names in the quotations on this page is intentional. If you are unable to understand why, please consult the Web.]

Page 820 in Table 3 (the value of B_1) ——— 21 Mar 2021

$-1/2$ ⇝ $1/2$

▸**Page 822** line 17 ——— 05 Jun 2019

in a previous volume ⇝ in another volume

Page 823 new entry to precede '$G\setminus v$' ——— 13 Apr 2012

$G\setminus U$	G with vertices of the set U removed	13

▸**Page 824** line 14 ——— 10 Jan 2011

$\binom{n}{k}$ ⇝ $\binom{x}{k}$

Page 824 line 24 ——— 13 Sep 2013

$\sim\!p$ or $\overline{p}$ ⇝ $\neg p$ or $\sim\!p$ or $\overline{p}$

Page 825 new entry to follow the ruler function ——— 07 Feb 2011

νn	sideways sum (when $n \ge 0$): $\sum_{k\ge 0}\bigl((n \gg k)\ \&\ 1\bigr)$	143

Page 825 Bernoulli number entry ——— 21 Mar 2021

$(e^z - 1)$ ⇝ $(1 - e^{-z})$

Page 826 new entry to follow 'complement of graph' ——— 03 Dec 2021

G^T	converse of digraph G (change '$\longrightarrow$' to '$\longleftarrow$')	§7.2.2.3

Page 826 new entry to follow 'elementary family' ——— 13 Apr 2012

$\wp$	universal family: all subsets of a given universe	275

▸**Page 826** line -6 or -7 (for the minimal elements) ——— 15 Jul 2013

$\mathcal{F}^{\uparrow}$ ⇝ $\mathcal{F}^{\downarrow}$

Page 827 new entries for the index to notations ——— 10 Nov 2021

$\alpha(G)$	independence number of G	35
$\gamma(G)$	domination number of G	673
$\kappa(G)$	vertex connectivity of G	§7.4.1.3
$\lambda(G)$	edge connectivity of G	§7.4.1.3
$\nu(G)$	matching number of G	§7.5.5
$\chi(G)$	chromatic number of G	35
$\omega(G)$	clique number of G	35
$c(G)$	number of spanning trees of G	482

Page 827 new quotation for bottom of page ——— 18 Jul 2019

A reader who has temporarily forgotten the definition of, say, "z"
in an argument can at least guess that it should be a complex number.
— TERENCE TAO, *Google Buzz feed* (20 March 2010)

▸**Page 830** lines 9 and 10 after the title ——— 31 Jan 2016

Boolean function of mn Boolean variables, with 0 representing TRUE and 1 representing FALSE ⤳ Boolean function of its index variables (i, j), with 0 representing FALSE and 1 representing TRUE

▸**Page 830** line −15 ——— 29 Oct 2021

negated literal $\bar{x}_j$ ⤳ nonnegated literal x_j.

Page 831 line 3 of the paragraph for Covering ——— 29 Jun 2017

The goal ⤳ Equivalently, find an implicant of a monotone Boolean function, given its clauses. The goal

Page 834 and following ——— 01 Jan 2011

Miscellaneous changes to the existing index of Volume 4A are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.