☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ ERRATA TO VOLUME 3 (after 2021) ☆☆☆☆

This document is a transcript of the notes that I have been making in my personal copy of *The Art of Computer Programming*, Volume 3 (second edition, 45th printing) since it was first printed in 2022. Previous errata are recorded in another file '`all3-pre.ps`'.

Four levels of updates — "errors," "amendments," "plans," and "improvements" — appear, indicated by four different typographic conventions:

▸**Page 666** line 1 ______ 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \rightsquigarrow n+1$

Page 666 line 2 ______ 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . .' leading to the date on which I recorded the plan in my files.

Page 666 line 4 ______ 10 Jan 1938

The fourth and final category — indicated by page and line number in smaller, slanted type — consists of minor corrections or improvements that most readers don't want to know about, because they are so trivial. You wouldn't even be seeing these items if you hadn't specifically chosen to print the complete errata list in all its gory details. Are you sure you wanted to do that?

My shelves at home are bursting with preprints and reprints of significant research results that I want to digest and summarize, where appropriate, in the ultimate edition of Volume 3. I didn't do that in the second edition because I would surely have to do it over again later: New results continue to pour forth at a great rate, and I will have time to rewrite that volume only once. Volumes 4 and 5 need to be finished first. So I've put most of my effort so far into writing up those parts of the total picture that seem to have converged to their near-final form. It follows, somewhat paradoxically, that the updates in this document are most current in the areas where there has been least activity.

On the other hand I do believe that the changes listed here bring Volume 3 completely up to date in two respects: (1) All of the research problems in the previous edition — i.e., all exercises that were rated 46 and above — have received new ratings of 45 or less whenever I learned of a solution; and in such cases, the answer now refers to that solution. (2) All of the historical information about pioneering developments has been amended whenever new details have come to my attention.

The ultimate, glorious, 100% perfect editions of Volumes 1–4A are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 4B, Stanford University, Stanford CA 94305-9045, or by email to `taocp@cs.stanford.edu`*. (Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although I'm working full time on Volume 4B these days, I will try to reply to all such messages within a year of receipt. Current news about The Art of Computer Programming is posted on*

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

and updated regularly.

— Don Knuth, February 2022

Writing a series like The Art of Computer Programming
is similar to painting the Forth Rail Bridge.
No sooner is it finished than
the job must be started again.
— MALCOLM CLARK (1992)

The time when The Guardian *ceases to make mistakes altogether*
is not, at the moment, foreseeable.
— IAN MAYES (1998)

☆☆☆☆☆☆☆ SORTING AND SEARCHING ☆☆☆☆☆☆☆

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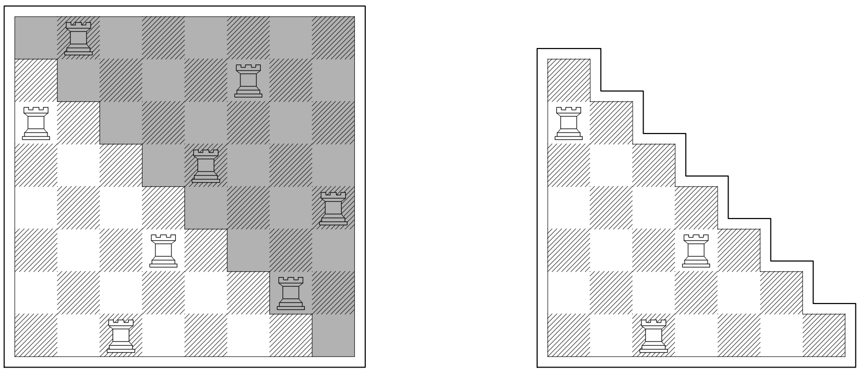
Last updated 12 August 2025

Most of these corrections have already been made in recent printings.

Page 46 hi-res replacement for Fig. 4 ——— 03 Feb 2025



Page 64 append to the paragraph following (53) ——— 16 Nov 2024

All terms up to $O(n^{-85})$ have been computed by S. B. Ekhad, M. Kauers, and D. Zeilberger (see arXiv:2410.16334 [math.CO] (2024), 4 pages).

▸**Page 93** in Table 5 for increments '4 2 1' ——— 15 Mar 2023

$274.42u \rightsquigarrow 274.41u$

▸**Page 93** replacement for lines −5 and −4 ——— 26 May 2025

These increments $(h_0, h_1, h_2, \ldots) = (1, 5, 19, 41, 109, 209, 505, \ldots)$ combine two sequences that resemble increments for which Sedgewick proved the worst-case time bound $O(N^{4/3})$.

▸**Page 95** lines −5 and −4 ——— 12 Mar 2023

using the quantity $\lfloor 2.25h_s \rfloor$ in place of $3h_s$ in (12); $\rightsquigarrow$
using the formula $h_s = \left\lfloor \frac{4}{5}\left(\frac{9}{4}^{s+1} - 1\right) \right\rfloor$;

Page 170 line 9 ——— 18 Jun 2025

(1929) $\rightsquigarrow$ (London, 1929–1930)

Page 179 new quotation for middle of page ——— 21 Jun 2023

To do this most quickly and arranged optimally, use this method.
Write down whatever is to appear in the index,
as soon as it presents itself, without any regard for order,
using one side of a sheet of paper, leaving the other side blank. . . .
Cut everything into pieces with scissors, and divide the slips
into the desired order, first into large parts,
then subdividing once or twice, or as often as needed. . . .

If there are many slips, I recommend subdividing more often:
all will then be easier to sort, with less confusion.
— CONRAD GESSNER, *Pandectæ* (1548)

Page 192 line 16 ———— 12 Aug 2022

Vychisl. Matem. i Matem. Fiziki **62** $\leadsto$ *Vychisl. Matem. i Matem. Fiziki* **61**

Page 197 new exercise ———— 20 Dec 2023

37. [*HM39*] (R. P. Stanley, 1981.) Let $A(G)$ be the $n\times n$ matrix for the graph G that's defined after (33). Prove that $A_{ik}^2 \ge A_{i(k-1)}A_{i(k+1)}$, for $1 \le i \le n$ and $1 < k < n$.

▸**Page 224** line 1 below Fig. 48 [only in 2021 and 2022 printings] ———— 17 Mar 2023

ge the sorted result $\leadsto$ get the sorted result

Page 246 new rating for exercise 67 ———— 22 Feb 2024

[*M48*] $\leadsto$ [*M46*]

▸**Page 369** line −19 ———— 04 Feb 2023

need to replenished $\leadsto$ need to be replenished

▸**Page 379** line 3 for exercise 27 ———— 10 Apr 2023

we throw m_k balls into urns $(X_k+j) \bmod n$ $\leadsto$ we throw a ball into urn $(X_k+j) \bmod n$

▸**Page 400** line 9 ———— 15 Nov 2023

Reading, Mass. $\leadsto$ Cambridge, Mass.

Page 451 line 2 of step G4 ———— 30 Sep 2024

Set l_k to the distance $\leadsto$ Set $l_k = \mathtt{LEVEL(X}_k\mathtt{)}$ to the distance

Page 457 new rating for exercise 29 ———— 11 Nov 2023

[*40*] $\leadsto$ [*30*]

▸**Page 478** line −14 ———— 26 May 2023

540–546 $\leadsto$ 540–545

Page 480 append a sentence to exercise 21 ———— 30 Sep 2024

Assume that $N > 0$.

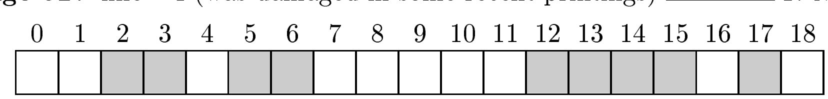
Page 493 and page 494 in lines 05, 06, 08 of Program P ———— 30 Sep 2024

`Q` $\leadsto$ `X`

▸**Page 518** line 12 ———— 01 Nov 2022

461–471 $\leadsto$ 461–472

▸**Page 527** line −4 (was damaged in some recent printings) ———— 27 Nov 2024



(21)

Page 531 the line before (31) ———— 03 Nov 2024

M is prime greater than $\leadsto$ M is prime and greater than

Page 534 line 2 — 06 Feb 2022

the algorithm terminates. $\rightsquigarrow$ set $N \leftarrow N - 1$ and terminate the algorithm.

Page 562 lines −11 and following — 03 Nov 2024

30,000 bits, so an inverted list ... shortest possible one.)$\rightsquigarrow$
30,000 bits. An inverted list for any particular attribute will occupy at most a million bits; so it will fit in at most 34 tracks.

Page 584 new answer for Notes on the Exercises — 24 Nov 2023

2. They often extend the material that's in the main text.

Page 594 amplify line 10 of answer 16 — 26 Sep 2024

We get further insights from q-generalizations of factorials and exponentials. Let

$$n!_q = \prod_{k=1}^{n} (1 + q + \cdots + q^{k-1}); \qquad \exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!_q}.$$

Page 595 and top of page 596 — 22 Sep 2024

[remove the paragraph 'This exercise ... §7.]'

Page 610 first two lines of answer 22 — 22 Sep 2024

Assume that ... answer is $\rightsquigarrow$ The number of ways is clearly zero if $m > N$ and $n_m > 0$. Otherwise we can assume that $m = N$, by adding 0s to the shape if necessary. Then the answer is

▸**Page 612** line −5 of answer 33 — 10 Nov 2022

40–70 $\rightsquigarrow$ 49–70

Page 615 line −2 of answer 45 — 10 Nov 2022

A114 (2007) $\rightsquigarrow$ **A114** (2007), 211–234

▸**Page 618** correction to the Algorithm D′ of 21 December 2019 — 03 Sep 2024

Otherwise if $\texttt{COUNT}[K_r] = r$, set $\texttt{COUNT}[K_r] \leftarrow r$ $\rightsquigarrow$ Otherwise if $\texttt{COUNT}[K_r] = r$, set $\texttt{COUNT}[K_r] \leftarrow r - 1$

▸**Page 642** in answer 16 — 22 May 2022

line 7: $A_n = \lfloor \lg n \rfloor / n +$ $\rightsquigarrow$ $A_1 = 0$ and $A_n = \lfloor \lg n \rfloor / n +$
line 9: $(2^{l+1} - 2)$ $\rightsquigarrow$ $(2^{l+1} - l - 2)$

Page 649 line −10 of answer 18 — 22 Sep 2024

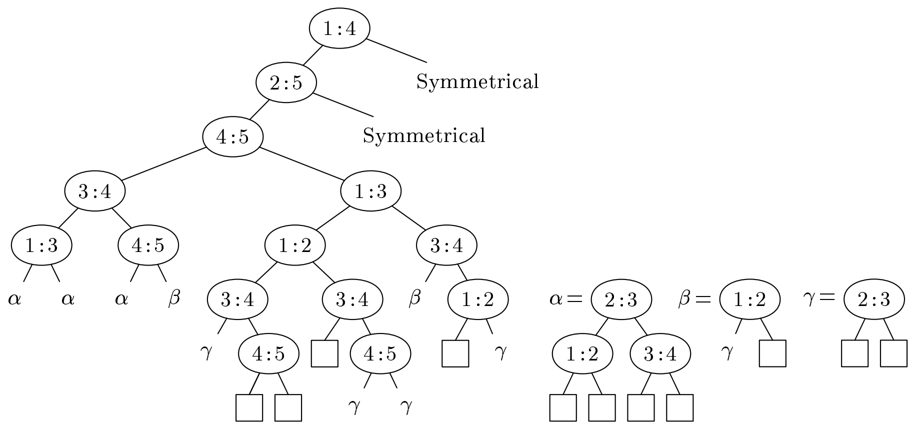
Since no element $\rightsquigarrow$ Since no key

Page 657 line 2 of answer 28 — 11 Nov 2023

optimal $\rightsquigarrow$ optimum

▸**Page 657** replacement for lines 4 and following of answer 31 ________ 12 Nov 2023

sorted in 3 more comparison-exchange operations. On the other hand the following tree, constructed by Filip Stappers in 2023, achieves the desired bound (and perhaps also the minimum *average* number of comparison-exchanges, namely $840/120 = 7$):



Page 657 new answer ________ 11 Nov 2023

32. Filip Stappers has shown that $S_e(n) = S(n)$ for $n = 6$ and $n = 7$.

Page 658 new answer ________ 20 Dec 2023

37. [The rows of $A(G)$ are said to be "log-concave." An algebraic proof appears in *J. Combinatorial Theory* **A31** (1981), 56–65; no purely combinatorial proof is known.]

▸**Page 659** in answer 12 ________ 01 May 2023

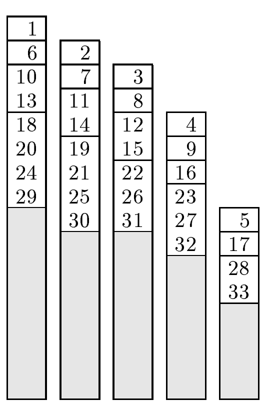
line 2: $\beta : X_{n-k} \rightsquigarrow \beta : X_{n+1-k}$
line 4: $Y_r = X_{r-k} \rightsquigarrow Y_r = X_{r+k}$

▸**Page 660** new answer (and remove this citation from answer 22) ________ 21 Mar 2023

18. See Ľ. Kollár, *Computers and Artificial Intelligence* **5** (1986), 335–344.

▸**Page 680** bottom illustration (was damaged in some recent printings) ____ 27 Nov 2024

1.



▸**Page 690** line 7 of answer 25 ——— 15 Oct 2024

exercise 22(c) $\leadsto$ exercise 23(c)

Page 697 end of answer 15 ——— 15 Oct 2024

after k stops $\leadsto$ if we need k stops

▸**Page 711** replacement for answer 29 ——— 11 Nov 2023

29. It is a degenerate tree (see exercise 5) with `THIS` at the top, `THE` at the bottom, needing 22.907 comparisons for an average successful search.

Douglas A. Hamilton has proved that some degenerate tree is always worst. Therefore, by exercise 6.1–14, we can find pessimal binary search trees in $O(n \log n)$ steps.

Page 713 line 8 of answer 44 ——— 30 Sep 2024

The variable j $\leadsto$ Optionally, also set $\mathtt{WT(X}_j\mathtt{)} \leftarrow \mathtt{WT(LLINK(X}_j\mathtt{))} + \mathtt{WT(RLINK(X}_j\mathtt{))}$. The variable j

▸**Page 719** in answers 1 and 2 ——— 01 May 2023

$_{223}$ $\leadsto$ $_{233}$ (in the root node)

Page 759 and following ——— 01 Feb 2022

Miscellaneous changes to the existing index of Volume 3 are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.