☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ ERRATA TO VOLUME 3 (after 2010) ☆☆☆☆

This document is a transcript of the notes that I had been making in my personal copy of *The Art of Computer Programming*, Volume 3 (second edition, 27th printing), from the time it was printed in 2011 until the 45th printing came out in 2022. Previous errata are recorded in another file '`all3-prepre.ps`'.

Four levels of updates — "errors," "amendments," "plans," and "improvements" — appear, indicated by four different typographic conventions:

▸**Page 666** line 1 ________ 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \rightsquigarrow n+1$

Page 666 line 2 ________ 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . .' leading to the date on which I recorded the plan in my files.

Page 666 line 4 ________ 10 Jan 1938

The fourth and final category — indicated by page and line number in smaller, slanted type — consists of minor corrections or improvements that most readers don't want to know about, because they are so trivial. You wouldn't even be seeing these items if you hadn't specifically chosen to print the complete errata list in all its gory details. Are you sure you wanted to do that?

My shelves at home are bursting with preprints and reprints of significant research results that I want to digest and summarize, where appropriate, in the ultimate edition of Volume 3. I didn't do that in the second edition because I would surely have to do it over again later: New results continue to pour forth at a great rate, and I will have time to rewrite that volume only once. Volumes 4 and 5 need to be finished first. So I've put most of my effort so far into writing up those parts of the total picture that seem to have converged to their near-final form. It follows, somewhat paradoxically, that the updates in this document are most current in the areas where there has been least activity.

On the other hand I do believe that the changes listed here bring Volume 3 completely up to date in two respects: (1) All of the research problems in the previous edition — i.e., all exercises that were rated 46 and above — have received new ratings of 45 or less whenever I learned of a solution; and in such cases, the answer now refers to that solution. (2) All of the historical information about pioneering developments has been amended whenever new details have come to my attention.

The ultimate, glorious, 100% perfect editions of Volumes 1–4A are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 4B, Stanford University, Stanford CA 94305-9045, or by email to `taocp@cs.stanford.edu`. *(Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although I'm working full time on Volume 4B these days, I will try to reply to all such messages within a year of receipt. Current news about The Art of Computer Programming is posted on*

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

and updated regularly.

— Don Knuth, February 1998

Writing a series like The Art of Computer Programming
is similar to painting the Forth Rail Bridge.
No sooner is it finished than
the job must be started again.
— MALCOLM CLARK (1992)

The time when The Guardian *ceases to make mistakes altogether*
is not, at the moment, foreseeable.
— IAN MAYES (1998)

☆☆☆☆☆☆☆ SORTING AND SEARCHING ☆☆☆☆☆☆☆

Last updated 16 August 2022

All of these corrections have been made in subsequent printings.

▸**Page 0** (on the back page of the dustcover, line 23) ——— 10 Sep 2012

Whatever your backgound, ⇝ Whatever your background,

Page v make the quote match the 1624 punctuation and spelling ——— 22 May 2021

Cookery is become an art,
a noble science,
Cookes are Gentlemen.
— TITUS LIVIUS, *Ab Urbe Condita* XXXIX.vi
(Robert Burton, *Anatomy of Melancholy* 1.2.2.2, 1624)

Page vii line 6 before the author's signature ——— 23 Sep 2015

nobody noticed ⇝ nobody else noticed

Page x line 16 ——— 16 Jun 2011

a *45* rating ⇝ a *40* rating

Page x line 20 ——— 30 Jan 2013

creativity. ⇝ creativity. All exercises with ratings of *46* or more are open problems for future research, rated according to the number of different attacks that they've resisted so far.

▸**Page xii** line 8 ——— 04 Apr 2012

Internal sorting ⇝ Internal Sorting

Page 8 line 11 ——— 20 Jul 2019

McCall's cookbook ⇝ McCall's cook book

Page 25 line 3 ——— 01 Oct 2021

letters in common ⇝ elements in common

Page 34 in exercise 22 ——— 13 Apr 2012

line 6: 0^k ⇝ 0^p (twice)
line 6: $k \ge 0$ ⇝ $p \ge 0$

line −3: $\displaystyle\sum_{k\ge0}\frac{k!^2(n_{\le t}-k)!\,(n_{>t}-k)!}{n_1!\ldots n_m!}$ ⇝ $\displaystyle\sum_{p\ge0}\frac{p!^2(n_{\le t}-p)!\,(n_{>t}-p)!}{n_1!\ldots n_m!}$

line −1: $p_{\le t}=p_{>t}=k$ ⇝ $p_{\le t}=p_{>t}=p$

Page 42 lines −9 and −8 ——— 20 Apr 2019

Simon Newcomb's problem . . . a multiset. ⇝ Thus Simon Newcomb's problem was to find the probability distribution of runs in a random permutation of a certain multiset.

▸**Page 50** line 12 — 21 Apr 2019

current values of j, x_i, $\leadsto$ current values of i, j, x_i,

Page 68 line 4 of exercise 23 — 10 May 2020

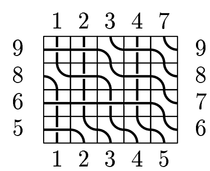
order? $\leadsto$ order? (These are called *alternating* or *up-down* permutations.)

Page 72 new exercise — 11 Aug 2020

▸**45.** [*30*] (E. Steingrímsson and L. K. Williams, 2007.) Take any array T of shape $(n_1, \ldots, n_m)$, where $n_1 \geq \cdots \geq n_m \geq 0$, and fill it with 0s and 1s so that (i) every column contains at least one 1; (ii) no 0 is simultaneously below and to the right of a 1. Let P be the n-step path from lower left to upper right that defines T's shape, where $n = m + n_1$. Write r_j above column j, where $r_1 < \cdots < r_{n_1}$ are the rightward steps of P; also write u_i at the left of row i, where $u_m < \cdots < u_1 = n$ are the upward steps. Replace every $\boxed{0}$ by and every $\boxed{1}$ by ; and add extra tiles , if necessary, to completely fill an $m \times n_1$ rectangle. Finally, write j at the bottom of column j and $n+1-i$ at the right of row i. The resulting array gives n disjoint paths, matching the numbers k at the left and top with corresponding numbers p_k at the right and bottom.

For example, one possibility for $(n_1, \ldots, n_m) = (5, 5, 4, 4)$ is shown below, together with its decorated tiling. None of the 1s in rows 2, 3, 4 of this array could be changed to 0 without violating (i) or (ii). The resulting permutation $p_1 p_2 \ldots p_9$ is 438526917.

$$T = \begin{array}{|c|c|c|c|c|}\hline 0&0&1&0&1\\ \hline 1&0&1&0&1\\ \hline 0&0&0&0\\ \cline{1-4} 0&1&1&1\\ \cline{1-4}\end{array}\;;$$



a) Index k of a permutation is called an *excedance* if $p_k > k$. Prove that after this construction the excedances of $p_1 p_2 \ldots p_n$ are precisely the indices $\{r_1, \ldots, r_{n_1}\}$.
b) What array gives a permutation $p_1 p_2 \ldots p_n$ with *no* excedances?
c) Suppose T yields a permutation $p_1 \ldots p_n$ where $p_k = k$ is a fixed point. What array yields the permutation $p_1{}' \ldots p'_{k-1} p'_{k+1} \ldots p_n{}'$, where $p' = p - [p \geq k]$?
d) Suppose T yields a permutation $p_1 \ldots p_n$ with no fixed points, and let k be its largest excedance. What array yields the permutation $p_1{}' \ldots p'_{k-1} p'_{k+1} \ldots p_n{}'$, where $p' = p - [p \geq p_k]$?
e) Explain how to obtain *any* desired permutation $p_1 \ldots p_n$ from some array T.
f) How many permutations $p_1 \ldots p_n$ have exactly r excedances? *Hint:* See 1.3.3–(20).

Page 86 line 12 — 07 Mar 2015

h_{t-2} sorting $\leadsto$ h_{t-2}-sorting

▸**Page 88** line 3 — 07 Mar 2015

$0.15n^{3/2} \leadsto 0.16n^{3/2}$

Page 96 steps L3 and L4 — 10 Mar 2015

record $R \leadsto$ record R_j (twice)

▸**Page 102** last line of exercise 6 — 06 Mar 2015

for $N + 1$ records $\leadsto$ for N records

Page 110 new sentence in paragraph before line −3 ——— 18 Aug 2021

Modern computers have however led to its rebirth, for sorting short files in a cache-efficient manner, without expensive branch-prediction errors.

▸**Page 111** in Fig. 17 on the arrow M3 → M4 ——— 07 Jun 2012

$i \wedge p = r$ ⇝ $i \mathbin{\&} p = r$

Page 129 in (33) ——— 22 Mar 2021

B_j ⇝ $(-1)^j B_j$

Page 129 in (36) ——— 22 Mar 2021

$\sum\limits_{0 \le t < m}$ ⇝ $\sum\limits_{0 < t \le m}$

Page 129 line −5 ——— 16 Jul 2011

used before ⇝ used at the end of Section 5.1.4

Page 133 the line following (48) ——— 05 Nov 2011

ignore $\delta(n)$ ⇝ ignore the "wobbles" of $\delta(n)$

Page 136 in exercise 35 ——— 22 Apr 2015

frequencies ⇝ quantities

▸**Page 137** in exercise 49 ——— 23 Apr 2015

(47) ⇝ (50)

▸**Page 174** line *20* of the `MIX` program ——— 04 Feb 2012

To R3 if end of pass. ⇝ To R3 if not end of pass.

Page 192 new paragraph to follow line 13 ——— 16 Jan 2022

Such results gave $S(n) \le \lg n! + cn + O(\log n)$ for $c > 0$. An asymptotic breakthrough was achieved by I. S. Sergeev [*Zhurnal Vychisl. Matem. i Matem. Fiziki* **62**, 2 (2021), 345–362], who proved that $S(n) \le \lg n! + O(n/(\log n)^{1/5})$.

Page 202 improvement to proof of Theorem K ——— 16 May 2015

Line 3: large. We ⇝ large. Let d be 2, 3, or 4. We
Line 6: on d, for $d \le 4$. ⇝ on d.
Line 8: on m. ⇝ on m. The upper bound follows from (4) and (8).

▸**Page 202** line −16 ——— 16 May 2011

F. Hwang ⇝ F. K. Hwang

▸**Page 205** line 1 of exercise 11 ——— 16 May 2011

F. Hwang ⇝ F. K. Hwang

Page 210 in (6) ——— 20 Jan 2013

$n \ge t$ ⇝ $1 \le t \le n$

Page 213 line 2 before (14) ________ 08 Oct 2013

Improved $\rightsquigarrow$ Then Kirkpatrick discovered that $V_3(42) = 50$; this may well be the only other counterexample [see *Lecture Notes in Comp. Sci.* **8066** (2013), 61–76]. Improved

Page 216 line 3 before the exercises ________ 16 May 2015

$\Theta(\lg(n!/T(G)) + n - k) \rightsquigarrow \Theta(\log(n!/T(G)) + n - k)$

Page 224 replacement for Fig. 48 and for lines 4–5 following (2) ________ 01 May 2021

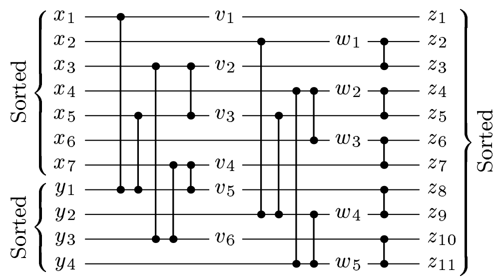


Fig. 48. The odd-even merge, when $m = 7$ and $n = 4$.

Batcher's (m, n)-merging network is called the *odd-even merge*. A $(7, 4)$-merge constructed according to these principles is illustrated in Fig. 48. (Notice that w_4 is below v_5 and w_5 is below $v_6 = v^*$ in this example; but no harm is done.)

Page 226 lines 5 and 6 after (11) ________ 12 Aug 2016

172]; the values of $\hat{S}(n)$ for $n > 8$ are still $\rightsquigarrow$ 172]; M. Codish, L. Cruz-Filipe, M. Frank, and P. Schneider-Kamp [*Journal of Computer and System Sciences* **82** (2016), 551–563] have also verified this when $n \le 10$. The remaining values of $\hat{S}(n)$ are still

Page 226 amendment to that amendment ________ 22 Jun 2020

The remaining $\rightsquigarrow$
J. Harder settled the cases $n = 11$ and $n = 12$ in arXiv:2012.04400 [cs.DS] (2020), 54 pages. The remaining

Page 229 line −5 ________ 16 Nov 2013

For $n \le 10$ . . . (see exercise 4). $\rightsquigarrow$ In fact all of the values given here are known to be exact (see the answer to exercise 4).

Page 234 lines 5–7 before the exercises ________ 30 Nov 2014

Andrew Yao . . . as $n \to \infty$; $\rightsquigarrow$ Andrew Yao [*SICOMP* **9** (1980), 566–582] determined the asymptotic behavior of $\hat{U}_t(n)$ for fixed t, by showing that $\hat{U}_3(n) = 2n + \lg n + O(1)$ and $\hat{U}_t(n) = n\lceil \lg(t+1)\rceil + O\bigl((\log n)^{\lfloor \lg((t+1)/3)\rfloor}\bigr)$ as $n \to \infty$;

Page 240 line 4 of exercise 37 ________ 21 Apr 2019

performed alternatively $\rightsquigarrow$ performed in alternation

Page 241 exercise 44 — 01 May 2021

some $n > 8$ ⇝ some $n > 12$

Page 242 last line of exercise 52 — 08 Apr 2015

co-NP-complete ⇝ coNP-complete

Page 253 new sentence at end of the caption for Fig. 62 — 20 Jan 2013

There are $P = 12$ external nodes.

Page 253 lines 1 and 2 after the caption for Fig. 62 — 20 Apr 2019

consists of replacing ..., and changing ⇝ replaces ..., and changes

Page 261 line −2 and page 262 line 1 — 20 Jan 2013

X ⇝ K [6 times]

▸**Page 264** line 2 of exercise 17 — 20 Jan 2013

$K_1 \geq K_2 \geq \cdots \geq K_N$ ⇝ $K_1 > K_2 > \cdots > K_N$

Page 268 line 8 — 15 Jun 2015

[insert a dash as the Contents of T3 in Phase 1]

Page 297 line 13 — 07 Jun 2016

(see exercise 6). ⇝ (see exercise 6). Jost Bürgi had already discovered very similar ideas before 1592; see M. Folkerts, D. Launert, and A. Thom, *Historia Mathematica* **43** (2016), 133–147.

Page 305 lines −2 and −1 — 20 Apr 2019

Step number i ..., and changing ⇝ Then, at step number i in the tree's growth, choose distinct tape names B, B_1, B_2, ..., B_k, and change

▸**Page 315** bottom line [but only in 17th–28th printings!] — 27 Mar 2011

instead of as a ⇝ instead of as

Page 330 line 6 after the caption to Fig. 85 — 19 Aug 2015

Figs. 70, 74, ⇝ Figs. 70, 72, 74,

Page 339 line 12 — 26 Aug 2015

numbers with the most significant fields at ⇝ numbers in a year-month-day format, so that the most significant fields are at

▸**Page 342** line 3 of exercise 12 — 31 Jul 2019

will contains blocks ⇝ will contain blocks

Page 344 line −4 — 20 Apr 2019

consists of reading from tape D and writing ⇝ reads from tape D, and it writes

Page 380 lines 18–20 — 18 Aug 2021

has no apparent ... suitably ⇝ was once thought to have no redeeming features, since there was always a better way to do what it did; but modern computer caches actually implement it efficiently. And the same technique, suitably

Page 381 line −10 ——— 29 Dec 2016

combining one or more ⇝ combining two or more

Page 382 in footnote d to Table 5.5–1 ——— 16 Sep 2014

5.2.1–29. ⇝ 5.2.1–29; $N^{7/6}$ is not rigorous.

▸**Page 392** line −8 ——— 12 Nov 2018

sometime desirable ⇝ sometimes desirable

Page 410 append new line to step B2 ——— 03 Aug 2019

(*Note:* If N can be really huge, it's safer to compute $i \leftarrow l + \lfloor (u-l)/2 \rfloor$.)

Page 411 line *07* of Program B ——— 15 Dec 2014

rA $\leftarrow l + u$. ⇝ rA $\leftarrow l + u$. (Overflow cannot occur.)

Page 412 line 15 after Fig. 5 ——— 09 May 2019

by means of comparisons ⇝ by means of comparisons alone, as in Section 5.3.1,

▸**Page 422** line 25 ——— 05 Jan 2017

n is extremely large ⇝ N is extremely large

▸**Page 423** line 2 of exercise 8 ——— 27 Mar 2012

$\sum_{j=0}^{\lfloor \lg N \rfloor + 2}$ ⇝ $\sum_{j=1}^{\lfloor \lg N \rfloor + 2}$

▸**Page 426** line 6 of exercise 28 ——— 30 Oct 2012

$((x*x)*x)*((x*x)*x)$ ⇝ $(x*(x*x))*(x*(x*x))$

Page 432 line 11 ——— 20 Jul 2015

`Q` $\equiv$ `LLINK(P)` or `RLINK(P)` ⇝ `Q` $\equiv$ `LLINK(P)` or `Q` $\equiv$ `RLINK(P)`

Page 443 line 6 ——— 05 May 2016

$x \lg(1/x)$ is concave; ⇝ $x \lg(1/x)$ is concave, when $x > 0$;

Page 460 line 2 of Theorem A ——— 01 Jan 2022

$N + 2$ ⇝ $N + 1.063$

Page 460 replacement for line −8 ——— 01 Jan 2022

$$N \ge F_{h+2} - 1 = (\phi^{h+2} - \hat{\phi}^{h+2})/\sqrt{5} - 1 \ge \phi^{h+2}/\sqrt{5} - \phi^{-4}/\sqrt{5} - 1,$$

Page 464 line 16 ——— 03 Apr 2016

$C \log N$ ⇝ $C \lg N$

Page 486 line 11 ——— 15 May 2021

split per insertion ⇝ splits per insertion

Page 458 new exercise ——— 16 Sep 2021

▸**50.** [*30*] Let $p_1 p_2 \ldots p_n$ be a permutation of $\{1, 2, \ldots, n\}$. Suppose the values p_1, p_2, …, p_n have been inserted successively into an initially empty binary tree using Algorithm T, but with `Q` $\leftarrow K$ in step T5 when storing key K. Explain how to compute all of the resulting links `LLINK(`k`)` and `RLINK(`k`)` for $1 \le k \le n$ in just $O(n)$ steps. (For example, the permutation 3142 would yield (`LLINK(1)`, …, `LLINK(4)`) $= (\Lambda, \Lambda, 1, \Lambda)$ and (`RLINK(1)`, …, `RLINK(4)`) $= (2, \Lambda, 4, \Lambda)$.)

Page 489 line 21 ——— 04 Apr 2016

$\ln 2 = 69.3$ ⇝ $\ln 2$; that's about 69.3

Page 505 line −10 ——— 11 Jan 2021

Patrician trees ⇝ Patricia trees

Page 506 line 6 ——— 11 Jan 2021

Patrician trees ⇝ Patricia trees

Page 510 line 1 of exercise 30 ——— 11 Jan 2021

Patrician generating function ⇝ Patricia generating function

Page 510 line 1 of exercise 32 ——— 11 Jan 2021

random Patrician ⇝ random Patricia

Page 513 line −14 ——— 12 Jul 2012

Section 2.3 ⇝ Section II.3

Page 518 line 5 ——— 30 Aug 2017

long ago by botanists ⇝ long ago by

Page 534 line 4 ——— 01 Oct 2014

or if ⇝ or

Page 570 line −1 ——— 28 Dec 2017

Superimposed coding ⇝ Superimposed random coding

Page 571 line 8 ——— 28 Dec 2017

32. ⇝ 32; *U.S. Patent 3521034* (1970).

Page 571 line 9 ——— 28 Dec 2017

superimposed coding ⇝ superimposed random coding

▸**Page 573** line −16 ——— 21 Apr 2016

$286/8008 \approx 3.5$ percent ⇝ $286/8008 \approx 3.6$ percent

▸**Page 583** line 10 ——— 17 Feb 2019

Information and Control **63** ⇝ *Information and Control* **64**

Page 585 additional comments for the code in answer 8 ——— 14 Sep 2019

line 2 (after `SRAX 1`): `SRAX 5` would also work
line 5 (after `SLAX 5`): $\mathrm{sign}(a - b)$

Page 588 line 23, right column ——— 20 Jul 2019

`COOKBOOK` ⇝ `COOK␣8BOOK`

Page 589 lines 10 and 11 of answer 19 ——— 16 May 2015

suggested by Jiang Ling ⇝ suggested by Ling Jiang

Page 593 lines 1–4 — 21 Mar 2019

respective vectors ... as the difference $\rightsquigarrow$
respective vectors $(1,1,0)$, $(1,0,1)$, $(0,-1,1)$, $(0,1,1)$, $(-1,0,1)$, $(-1,1,0)$, divided by $\sqrt{2}$, stand for adjacent interchanges of the respective pairs 21, 31, 32, 41, 42, 43. The sum of these vectors gives $(0,2,4)/\sqrt{2}$ as the difference

▸**Page 595** line 2 of answer 23 — 31 Mar 2019

$1+2k_2 \rightsquigarrow 1+2k_4$

Page 597 line 2 of answer 28 — 5 Oct 2015

hence td $\rightsquigarrow$ hence we have total displacement td

Page 598 line 12 of answer 10 — 13 Apr 2012

for some $k>1$ $\rightsquigarrow$ for some smallest $k>1$

Page 600 line 7 of answer 18 — 10 Oct 2015

giving all elements $\rightsquigarrow$ giving each of the n_j elements

Page 601 in answer 22 — 13 Apr 2012

line 1: $0^k \rightsquigarrow 0^p$
line 1: for some k $\rightsquigarrow$ for some p
line 3: the number k $\rightsquigarrow$ the number p
line −4: $0^k \rightsquigarrow 0^p$
line −3: $k!\,(n_1+\cdots+n_t-k)! \rightsquigarrow p!\,(n_1+\cdots+n_t-p)!$
line −3: ${\scriptstyle p_1+\cdots+p_t=k} \rightsquigarrow {\scriptstyle p_1+\cdots+p_t=p}$
line −2: $k=0 \rightsquigarrow p=0$

Page 602 replacement for first sentence of answer 3 — 22 Mar 2021

3. $\displaystyle\sum_n \frac{x^n}{n!}\sum_k \left\langle {n \atop k} \right\rangle (-1)^k = \frac{2}{e^{-2x}+1} = \frac{1}{x}\left(\frac{4x}{1-e^{-4x}} - \frac{2x}{1-e^{-2x}}\right)$ by (20), hence the result is $B_{n+1}2^{n+1}(2^{n+1}-1)/(n+1)$.

Page 602 last line of answer 3 — 22 Mar 2021

$2B_{n+1}(1-2^{n+1} \rightsquigarrow 2(-1)^n B_{n+1}(2^{n+1}-1)/(n+1)$

Page 610 line 1 of answer 23 — 28 Jul 2012

Journal $\rightsquigarrow$ *Comptes Rendus Acad. Sci.* **88** (Paris, 1879), 965–967; *Journal*

Page 611 new line at end of answer 23 — 29 May 2020

Row n shows how many up-down permutations end with 1, 2, ..., n, respectively.

Page 611 last line of answer 30 — 03 Mar 2021

paper #R15 $\rightsquigarrow$ #R15, 1–32, §2

Page 614 new sentence appended to answer 40 — 06 Apr 2014

See also Dan Romik, *The Surprising Mathematics of Longest Increasing Subsequences* (2014), Chapter 4.

Page 615 new answer ——— 11 Aug 2020

45. (a) Induction on the number of 0s. If we change the "last" 0 to 1, we either change $p_k \leftarrow p_k + 1$ for an excedance k, or $p_k \leftarrow p_k - 1$ for a nonexcedance k.

(b) The identity permutation $12 \ldots n$ is obtained when $n_1 = \cdots = n_m = 0$, $m = n$.

(c) The row with $u_i = k$ is all 0s in T. Remove row i by deleting n_i from the shape.

(d) We have $r_{n_1} = k$, and column n_1 has $n - p_k$ 0s followed by all 1s. Remove column n_1 by setting $n_i \leftarrow n_1 - 1$ whenever $n_i = n_1$.

(e) We know T's shape from (a). Now work backwards to fill in the cells, repeatedly removing fixed points via (c) and the largest remaining excedance via (d).

(f) It's the Eulerian number $\left\langle {n \atop r} \right\rangle$: Let $q_1 \ldots q_n$ correspond to $p_1 \ldots p_n$ as in the hint. Every ascent $q_i < q_{i+1}$ corresponds to an excedance $p_{q_i} = q_{i+1}$, and vice versa.

[See *Journal of Combinatorial Theory* **A114** (2007), for many further results. See also A. Postnikov, arXiv:math/0609764 [math.CO] (2006), 79 pages, §19.]

▸**Page 617** line −5 of answer 10 ——— 29 Oct 2020

I. D. G. MacLeod $\leadsto$ I. D. G. Macleod

Page 618 replacement for first lines of answer 13 ——— 21 Dec 2019

13. D5′. [Start final shuffle.] (All `COUNT`s are as in D5.) Set $r \leftarrow N$.

D6′. [Move R_r right?] If $r = 0$, stop. Otherwise if `COUNT[`K_r`]` $> r$, set $R \leftarrow R_r$, $j \leftarrow$ `COUNT[`K_r`]`, `COUNT[`K_r`]` $\leftarrow j - 1$. Otherwise if `COUNT[`K_r`]` $= r$, set `COUNT[`K_r`]` $\leftarrow r$, $r \leftarrow r - 1$, and repeat this step. Otherwise just set $r \leftarrow r - 1$ and repeat this step.

D7′. [Permute a cycle.] Set $S \leftarrow R_j$, $k \leftarrow$ `COUNT[`K_j`]`, `COUNT[`K_j`]` $\leftarrow k - 1$, $R_j \leftarrow R$, $R \leftarrow S$, $j \leftarrow k$. Then if $j \neq r$ repeat this step; but if $j = r$ set $R_j \leftarrow R$, $r \leftarrow r - 1$, and go back to D6′. ▮

Page 620 in answer 10 ——— 29 Apr 2013

line 2 after the program: time savings $\leadsto$ time saved
lines 5–6 after the program: not desirable ... unless $\leadsto$ usually insignificant, since only $O(\log N)$ time is saved in that case unless

▸**Page 621** replacement for line 7 of answer 16 ——— 17 Sep 2010

$$a_{i+jh} = 1 + q(h - i) + (r - i)[i \le r] + j \qquad \text{for } 1 \le i \le h \text{ and } j \ge 0$$

▸**Page 629** line *23* of the program in answer 12 ——— 06 Jun 2012

`J4N 4B` $\leadsto$ `J4N 8B`

Page 630 append a new line to answer 15 ——— 01 May 2021

[The function $c(n)$ defined in Eq. 5.3.4–(10) turns out to be identical to this one.]

Page 631 line 6 of answer 18 ——— 11 Mar 2016

line 6: "$\le$," $\leadsto$ "$\le$",

▸**Page 632** line 1 ——— 20 Apr 2016

M. Regnier $\leadsto$ M. Régnier

Page 636 line 1 of answer 44 ——— 22 Mar 2021

$-\delta_{k0}$ $\leadsto$

Page 637 line 7 of answer 51 ——— 22 Mar 2021

$\zeta(-n) = (-1)^n B_{n+1}/(n+1) \leadsto \zeta(-n) = -B_{n+1}/(n+1)$

▶**Page 642** line 6 of answer 16 ——— 18 Mar 2016

step 4 $\leadsto$ step I4

▶**Page 644** line 2 of answer 28 ——— 24 Apr 2015

$21\frac{2}{3}N \log N \leadsto 21\frac{2}{3}N \log_3 N$

▶**Page 660** line 9 of answer 23 ——— 26 Feb 2011

$x_{11} \leftarrow 1 \leadsto x_{11} \leftarrow 0$

▶**Page 664** line 9 ——— 31 Mar 2016

the first $t-1$ steps $\leadsto$ steps 1, 2, ..., $t-1$

Page 666 last three lines of answer 4 ——— 16 Nov 2013

Ian Parberry ...116. $\leadsto$ Further values have been obtained by D. Bundala and J. Závodný via satisfiability encoding (see Section 7.2.2.2). The value of $\hat{T}(17)$ remains unknown.

Page 667 line 7 of answer 13 ——— 21 Apr 2019

alternatively $\leadsto$ alternately

▶**Page 668** line 2 of answer 24 ——— 30 Nov 2014

by exercise 21 $\leadsto$ by exercise 22(a)

Page 669 line −3 of answer 27 ——— 18 Dec 2011

103–115. The fact $\leadsto$ 103–115; see also B. E. Tenner, *Annals of Combinatorics* **11** (2007), 101–114. The fact

Page 669 append to the end of answer 32 ——— 16 Dec 2015

[*Princeton Conference on Information Sciences and Systems* **6** (1972), 387–391.]

Page 669 line −3 of answer 35 ——— 02 Apr 2016

strike out $\leadsto$ remove

▶**Page 670** replacement for line 11 of answer 38 ——— 10 Jan 2019

$$P = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 2&3&4&5\\\cline{1-4} 3&4&5\\\cline{1-3} 4&5\\\cline{1-2} 5\\\cline{1-1}\end{array}\,, \qquad Q = \begin{array}{|c|c|c|c|c|}\hline 1&3&5&7&13\\\hline 2&6&8&15\\\cline{1-4} 4&9&12\\\cline{1-3} 10&11\\\cline{1-2} 14\\\cline{1-1}\end{array}\,.$$

Page 677 line 2 of answer 11 ——— 20 Jan 2013

sequence. $\leadsto$ sequence; hence the probability is 1/2, given that `LASTKEY` $\neq \infty$.

▶**Page 681** replacement for first two lines of answer 7 ——— 28 May 2018

7. Let $\alpha_p = 2x$ and $z = -1/2^{p+1}$. Then $x^{p+1} = x^p + z$; so exercise 1.2.6–25 gives the convergent series $\alpha_p = 2\sum_{k\ge 0}\binom{1-kp}{k} z^k/(1-kp) = 2 - 2^{-p} - p2^{-2p-1} + O(p^2 2^{-3p})$.

▸**Page 687** last line of answer 7 ———— 02 Jun 2013

polyphase's 176 ⇝ polyphase's 180

Page 713 new answer ———— 17 Sep 2021

50. See R. E. Tarjan's elegant bottom-up solution in the author's program OFFLINE-TREE-INSERTION (available online).

Page 720 top line ———— 20 Feb 2021

too empty ⇝ less than half full

Page 721 last line of answer 10 ———— 06 Feb 2021

1–21; ⇝ 1–21; P. L. Lehman and S. B. Yao, *ACM Trans. Database Systems* **6** (1981), 650–670;

▸**Page 723** line 3 of answer 10 ———— 15 May 2016

K is one less ⇝ k is one less

▸**Page 730** line 2 of step A7 ———— 02 Oct 2012

`TABLE[`i`]` ⇝ `TABLE[`j`]`

▸**Page 732** line 7 of answer 30 ———— 16 Apr 2016

$q_{j-k-1} < q_j$ ⇝ $q_{j-k+1} < q_j$

Page 734 line −2 ———— 19 Apr 2016

$1 - e^{-1} - e^{-2} = 0.4968$ ⇝ $1 - e^{-1} - e^{-2} \approx 0.4968$

Page 750 in Table 3 (the value of B_1) ———— 21 Mar 2021

$-1/2$ ⇝ $1/2$

Page 753 through page 755 ———— 07 Apr 2012

[replace the notation $(R \Longrightarrow x;\ y)$ by $(R?\ x\colon y)$ in eleven places]

Page 755 Bernoulli number entry ———— 21 Mar 2021

$(e^z - 1)$ ⇝ $(1 - e^{-z})$

Page 757 new entries for Appendix C ———— 17 Apr 2011

Program 5.2.1D′, 620.
Program 6.1S′, 702.

▸**Page 757** and 758 in Appendix C ———— 17 Apr 2011

Theorem 5.1.2C, 28–29.
Algorithm 5.2.2R, 123, 125.
Algorithm 5.4.2A, 267.
Program 6.2.3A, 464–466.
[also delete the first entry for Program 5.2C]

Page 759 and following ———— 01 Jan 2011

Miscellaneous changes to the existing index of Volume 3 are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.