☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ ERRATA TO VOLUME 1 (after 2010) ☆☆☆☆

This document is a transcript of the notes that I had been making in my personal copy of *The Art of Computer Programming*, Volume 1 (third edition, 27th printing), from the time it was first printed in 2011 until the 49th printing came out in 2022. Previous errata are recorded in another file '`all1-prepre.ps`'.

Four levels of updates — "errors," "amendments," "plans," and "improvements" — appear, indicated by four different typographic conventions:

▸**Page 666** line 1 ——————————————— 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \rightsquigarrow n+1$

Page 666 line 2 ——————————————— 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . .' leading to the date on which I recorded the plan in my files.

Page 666 line 4 ——————————————— 10 Jan 1938

The fourth and final category — indicated by page and line number in smaller, slanted type — consists of minor corrections or improvements that most readers don't want to know about, because they are so trivial. You wouldn't even be seeing these items if you hadn't specifically chosen to print the complete errata list in all its gory details. Are you sure you wanted to do that?

My shelves at home are bursting with preprints and reprints of significant research results that I want to digest and summarize, where appropriate, in the ultimate edition of Volume 1. I didn't do that in the third edition because I would surely have to do it over again later: New results continue to pour forth at a great rate, and I will have time to rewrite that volume only once. Volumes 4 and 5 need to be finished first. So I've put most of my effort so far into writing up those parts of the total picture that seem to have converged to their near-final form. It follows, somewhat paradoxically, that the updates in this document are most current in the areas where there has been least activity.

On the other hand I do believe that the changes listed here bring Volume 1 completely up to date in two respects: (1) All of the research problems in the previous edition — i.e., all exercises that were rated 46 and above — have received new ratings of 45 or less whenever I learned of a solution; and in such cases, the answer now refers to that solution. (2) All of the historical information about pioneering developments has been amended whenever new details have come to my attention.

The ultimate, glorious, 100% perfect editions of Volumes 1–4 are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 1B, Stanford University, Stanford CA 94305-9015, or by email to `taocp@cs.stanford.edu`*. (Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although I'm working full time on Volume 4 these days, I will try to reply to all such messages within a year of receipt. Current news about The Art of Computer Programming is posted on*

http://www-cs-faculty.stanford.edu/~knuth/taocp.html

and updated regularly.

— Don Knuth, January 2011

What happened?
The subject took the bit in its teeth and ran away with it,
that's what happened.
I know now how Sir James Frazer felt when,
after setting out to dash off a brief monograph
on a single obscure rite, he found himself
in the embarrassing possession of
the 12 volumes of "The Golden Bough."

— WAVERLEY ROOT (1974)

No matter how many times you proofread your book,
after publication there will be roughly one typo per page.

— JOSEPH H. SILVERMAN (2017)

☆☆☆☆☆ FUNDAMENTAL ALGORITHMS ☆☆☆☆☆

Last updated 16 August 2022

All of these corrections have been made in subsequent printings.

▸**Page 0** (on the back page of the dustcover, line 23) ——— 10 Sep 2012

Whatever your backgound, ⇝ Whatever your background,

▸**Page v** line 8 ——— 13 Dec 2018

McCall's Cookbook ⇝ *McCall's Cook Book*

▸**Page x** line 14 ——— 24 Mar 2016

American Mathematical Monthly ⇝ The American Mathematical Monthly

Page xi new paragraph to follow line 20 ——— 25 Nov 2013

My efforts to extend and enhance these volumes have been enormously enhanced since 1980 by the wise guidance of Addison–Wesley's editor Peter Gordon. He has become not only my "publishing partner" but also a close friend, while continually nudging me to move in fruitful directions. Indeed, my interactions with dozens of Addison–Wesley people during more than three decades have been much better than any author deserves. The tireless support of managing editor John Fuller, whose meticulous attention to detail has maintained the highest standards of production quality in spite of frequent updates, has been particularly praiseworthy.

Page xi line 5 before the author's signature ——— 23 Sep 2015

nobody noticed ⇝ nobody else noticed

Page xiv date of the Herbert quotation ——— 26 Nov 2019

(1640) ⇝ (1651)

Page xiv replacement for the Vauvenargues quotation ——— 21 Dec 2019

Les meilleurs auteurs parlent trop.

— VAUVENARGUES, *Paradoxes, mêlés de Réflexions et Maximes*, 115 (1746)

Page xvi line 16 ——— 16 Jun 2011

a *45* rating ⇝ a *40* rating

Page xvi line 20 ——— 30 Jan 2013

creativity. ⇝ creativity. All exercises with ratings of *46* or more are open problems for future research, rated according to the number of different attacks that they've resisted so far.

Page 1 the Epictetus quote ——— 16 Jan 2022

— EPICTETUS, *Discourses* IV.i (C. A.D. 110)

[also also move this quote up, above the one by Ada Augusta]

Page 2 line -7 — 13 Sep 2019

also usually appears ⇝ often appears also

Page 6 lines 1–3 — 13 Mar 2013

$\{m,n\}$ ⇝ m and n [twice]
$\{n,r\}$ ⇝ n and r [twice]

Page 7 line -5 — 22 Feb 2021

Q is a set containing subsets I and Ω, ⇝ Q is a set, I and Ω are subsets of Q,

Page 8 lines -5 *and* -4 — 13 Mar 2013

Such a computational method ... it is also ⇝ Every step of such a computational method is clearly effective, and experience shows that pattern-matching rules of this kind are also

Page 8 line -11 — 30 Jul 2019

for strings α and ω ⇝ for some strings α and ω in A^*.

Page 9 the rating of exercise 7 — 05 Apr 2014

M21 ⇝ *HM21*

Page 11 lines 3–4 — 06 May 2015

An expansion ... can ⇝ A more leisurely presentation of much of the following material, together with related concepts, can

Page 13 line 3 after (3) — 06 Feb 2021

$F_2 = 1 < 1.6 < \phi^1$ ⇝ $F_2 = 1 < \phi^1$

Page 15 line -13 — 20 Apr 2019

consists of proving ⇝ constructs a proof

▸**Page 17** line 26 — 15 Dec 2013

11 (1838) ⇝ **12** (1838)

Page 18 replacement for line 5 of exercise 2 — 19 Apr 2019

$$a^{(n+1)-1} = a^n = \frac{a^{n-1} \times a^{n-1}}{b^{n-1}} = \frac{1 \times 1}{1} = 1, \quad \text{where } b = a^{(n-2)/(n-1)};$$

Page 18 clarification in exercise 4 — 10 Feb 2013

ϕ^{n-2}. ⇝ ϕ^{n-2} for all positive integers n.

Page 19 clarification in exercise 5 line 1 — 05 Feb 2013

exact divisors ⇝ positive integer divisors

Page 19 additional clarification, exercise 5 line 3 — 20 Sep 2019

one or more prime numbers. ⇝ one or more prime numbers, namely in the form $n = p_1 \ldots p_k$ for some $k \geq 1$, where each of $p_1, \ldots, p_k$ is prime.

Page 19 in exercise 9 — 13 Sep 2019

$1 - na$ ⇝ $1 - na$ for all integers n

Page 22 lines 1 and 3 after (8) ——— 20 Jan 2013

$10^x \rightsquigarrow b^x$ [two places]

Page 22 line 3 after (8) ——— 17 Sep 2019

is 1.999... $\rightsquigarrow$ begins with 1.999...

Page 23 line 1 after (13) ——— 22 Feb 2015

Edward M. Reingold. $\rightsquigarrow$ Ernesto Trucco [*Bull. Math. Biophysics* **18** (1956), 130].

Page 25 lines 1 and 2 of exercise 11 ——— 22 Nov 2019

and $x \approx \log_{10} 2$, to how ... value of x in order to determine $\rightsquigarrow$ and x is approximately equal to $\log_{10} 2$, how many decimal places of x's value will guarantee that we can determine

Page 25 in exercise 18 ——— 22 Nov 2019

$\frac{1}{2} \lg x \rightsquigarrow \frac{1}{2} \lg x$, for all real $x > 0$.

▸**Page 27** line 1 ——— 22 Jul 2011

$n > 1 \rightsquigarrow n > 1$ and $n \neq e$.

Page 29 replacement for five lines after (9) ——— 22 Dec 2019

where $S'(j)$ specifies summing over all j such that "there is an integer i such that both $R(i)$ and $S(i,j)$ are true"; and $R'(i,j)$ specifies summing over all i such that "both $R(i)$ and $S(i,j)$ are true." For example, if the summation is $\sum_{i=1}^{n} \sum_{j=1}^{i} a_{ij}$, then $S'(j)$ sums over j where "there is an integer i such that $1 \le i \le n$ and $1 \le j \le i$," that is, $1 \le j \le n$; and $R'(i,j)$ sums over i where "$1 \le i \le n$ and $1 \le j \le i$," that is, $j \le i \le n$. Thus,

Page 30 last line of Example 1 ——— 20 Apr 2019

consists of simplifying $\rightsquigarrow$ simplifies

Page 35 replacement for line 2 of exercise 18 ——— 22 Dec 2019

$R(i)$ specifies summing over all i such that "n is a multiple of i" and $S(i,j)$ specifies summing over all j such that "$1 \le j < i$."

Page 37 line 10 after '**EXERCISES—Second Set**' ——— 14 May 2020

is $(-1)^{i+j}$ times $\rightsquigarrow$ is 1 if $n = 1$, otherwise it's $(-1)^{i+j}$ times

Page 37 in the three matrices displayed at the bottom ——— 22 Nov 2019

[insert vertical dots in the second column, to match those in the first and last columns]

▸**Page 38** last line of exercise 40 ——— 14 May 2020

1 when $j = n \rightsquigarrow (-1)^{n-1}$ when $j = n$

Page 45 line 2 of exercise 48 ——— 14 May 2020

$\left\lceil \frac{m}{n} \right\rceil \rightsquigarrow \left\lceil \frac{m}{n} \right\rceil$, if $n \neq 0$;

Page 45 new exercise ——— 03 Jun 2021

50. [*M20*] J. H. Quick noticed that $\lceil 1000\pi \rceil = 3142$ and $\lfloor 3142/\pi \rfloor = 1000$. Is it true that $\lceil m\pi \rceil = n \iff \lfloor n/\pi \rfloor = m$, for all integers m and n?

Page 47 replacement for displayed formula (7) ——— 22 Jan 2019

$$n! \approx \sqrt{2\pi}\left(\frac{n+1/2}{e}\right)^{n+1/2} \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{7}$$

▸**Page 50** line −5 ——— 15 Jun 2014

della Scienze ⇝ *delle Scienze*

Page 52 last line of exercise 24 ——— 22 Jun 2020

$k/(k-1)$.] ⇝ $k/(k-1)$ when $k > 1$.]

Page 53 line −3 thru line 1 of the text on page 54 ——— 19 Apr 2021

The earliest known . . . Another ⇝
Long before that (c. 700), in India, Virahāṅka had explained two ways to compute them in his *Vṛttajātisamuccaya* ("Manual of prosody"), slokas 6.7–6.11; we'll see those two ways below in Eqs. (9) and (11). Virahāṅka was inspired by an ancient Hindu classic, Piṅgala's *Chandaḥśāstra.* The earliest known detailed discussion of binomial coefficients is in Halāyudha's tenth-century commentary on Piṅgala's work; see G. Chakravarti, *Bull. Calcutta Math. Soc.* **24** (1932), 79–88. Another

Page 55 replacement for displayed equation (5) ——— 04 Oct 2021

$$\binom{r}{k} = \frac{r!}{k!\,(r-k)!}, \quad r \geq \text{integer } k \geq 0. \tag{5}$$

Page 56 in (10) ——— 04 Oct 2021

$n \geq 0.$ ⇝ $n.$

Page 56 line −8 ——— 04 Oct 2021

with two applications of Eq. (6) ⇝ and Eq. (6)

▸**Page 60** lines −3, −7, and −11 ——— 04 Oct 2021

$\sum_{k}$ ⇝ $\sum_{k\geq 0}$

▸**Page 61** lines 11, 15, and 16 ——— 04 Oct 2021

$\sum_{k}$ ⇝ $\sum_{k\geq 0}$

Page 61 replacement for the bottom 7 lines ——— 16 Nov 2016

to use Eq. (20) — which no longer applies.

In this situation it pays to complicate things even *more*, by replacing the unwanted $\binom{n+k}{m+2k}$ by a sum of terms of the form $\binom{x+k}{2k}$, since our problem will then become a sum of problems that we know how to solve. Accordingly, we use Eq. (25) with

$$k \mapsto j, \quad r \mapsto n-1, \quad m \mapsto m-1, \quad s \mapsto k, \quad n \mapsto 2k;$$

all of the conditions required for (25) are satisfied, because m and n are positive.

Page 62 replacement for the top 18 lines ——— 04 Oct 2021

The sum to be solved now takes the form

$$\sum_{k\geq 0}\sum_{j=0}^{n-1}\binom{n-1-j}{m-1}\binom{k+j}{2k}\binom{2k}{k}\frac{(-1)^k}{k+1}; \qquad (29)$$

and the rest is easy, by interchanging the order of summation: Equation (28) tells us how to sum on k, and all the complications melt away. Everything reduces to just

$$\sum_{j=0}^{n-1}\binom{n-1-j}{m-1}\delta_{j0},$$

so our final answer is

$$\binom{n-1}{m-1}.$$

The solution to this problem was fairly complicated, but not really mysterious; there was a good reason for each step. The derivation should be studied closely, because the use of (25) illustrates some delicate maneuvering with the conditions in our equations.

There is actually a better way to attack this problem, however! The reader is invited to figure out how to transform the given sum so that Eq. (26) applies (see exercise 30).

Page 63 insert a remark on line 4 ——— 04 Oct 2021

assuming, as we may, that $nt \neq r+s$.

▸**Page 63** lines −1, −2, −3, and −10 ——— 04 Oct 2021

$\sum_k \rightsquigarrow \sum_{k\geq 0}$

[and also insert 'when $m \geq 0$' on line −1]

Page 65 in (40) ——— 04 Oct 2021

insert 'integer $k \geq 0$,'

▸**Page 65** second line after (40) ——— 04 Oct 2021

each term $\rightsquigarrow$ each factor

Page 69 line 1 of exercise 10 ——— 04 Oct 2021

p is prime $\rightsquigarrow$ p is prime and n is an integer

Page 70 in exercise 25 ——— 07 Oct 2013

line 1: as in Eq. (30). $\rightsquigarrow$ as in Example 4 (see Eq. (30)).
line 5: the identity $\rightsquigarrow$ multiples of a special case of (34),

▸**Page 70** line 2 of exercise 25 ——— 07 Oct 2013

provided z is small enough $\rightsquigarrow$ provided that x is close enough to 1

Page 74 line 3 of exercise 67 ——— 04 Oct 2021

$n \geq k \geq 0 \rightsquigarrow n \geq k > 0$

Page 79 new exercise for Section 1.2.7 ——— 11 Feb 2013

25. [*M21*] Let $H_n^{(u,v)} = \sum_{1\le j\le k\le n} 1/(j^u k^v)$. What are $H_n^{(0,v)}$ and $H_n^{(u,0)}$? Prove the general identity $H_n^{(u,v)} + H_n^{(v,u)} = H_n^{(u)} H_n^{(v)} + H_n^{(u+v)}$.

▸**Page 79** line 3 after (2) ——— 24 Jan 2021

(Book of the Abacus) $\leadsto$ (Book of Calculation)

Page 80, line 6 ——— 19 Apr 2021

therefore $\leadsto$ and the rule for computing those numbers was stated by Virahāṅka (c. 700) in his work *Vṛttajātisamuccaya*, sloka 6.49.

Page 83, replacement for line −3 ——— 06 Feb 2021

$$F_n = \phi^n/\sqrt{5} \textit{ rounded to the nearest integer}, \quad \text{for all integers } n \ge 0. \qquad (15)$$

Page 85 the displayed formula in exercise 24 ——— 20 Jan 2017

insert 'det' before the matrix

Page 91 replacement for (29) ——— 21 Mar 2021

$$\frac{z}{1-e^{-z}} = 1 + \frac{1}{2}z + \frac{1}{12}z^2 + \cdots = \sum_{k\ge 0} \frac{B_k z^k}{k!}. \qquad (29)$$

▸**Page 100** line −5 ——— 24 Jul 2020

Roes *CACM* $\leadsto$ Roes, *CACM*

Page 104 line 11 ——— 26 Jun 2014

random variable $\leadsto$ random integer

Page 104 new sentence preceding the exercises ——— 19 Jul 2012

[S. Bernstein had contributed key ideas in *Uchenye zapiski Nauchno-Issledovatel'skikh kafedr Ukrainy* **1** (1924), 38–48.]

Page 108 replacement for bottom line ——— 20 Jan 2022

$$O\bigl(f(n)g(n)\bigr) = f(n)O\bigl(g(n)\bigr), \quad \text{if } f(n) \text{ is never zero.} \qquad (10)$$

Page 110 lines 14–16 ——— 01 Jul 2014

We have ... This equation $\leadsto$ We have

$$\sqrt[n]{n} = e^{\ln n/n} = 1 + (\ln n/n) + O\bigl((\log n/n)^2\bigr), \qquad (18)$$

because $\log n/n \to 0$ as $n \to \infty$; see exercises 8 and 11. (Notice that we needn't specify the base of 'log' inside O-notation, where constants are ignored.) This equation

Page 111 replacement for Fig. 12 and the lines just above and below 21 Mar 2021

Imperialis Petropolitanæ **6** (1732), 68–69; **8** (1735), 9–22.]

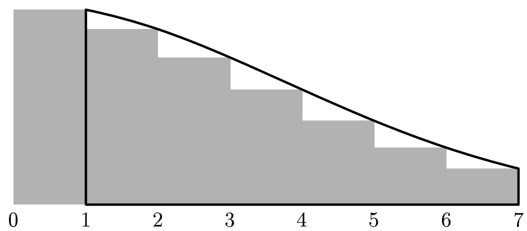


Fig. 12. Comparing a sum with an integral.

Figure 12 shows a comparison of $\int_1^n f(x)\,dx$ and $\sum_{k=1}^n f(k)$, when $n = 7$.

Page 112 replacement for (3) ——— 21 Mar 2021

$$\sum_{1\le k\le n} f(k) = \int_1^n f(x)\,dx + \tfrac{1}{2}\bigl(f(n)+f(1)\bigr) + \int_1^n B_1(\{x\})f'(x)\,dx, \qquad (3)$$

Page 112 replacement for (4) and the line above ——— 21 Mar 2021

are the coefficients in the following infinite series:*

$$\frac{z}{1-e^{-z}} = B_0 + B_1 z + \frac{B_2 z^2}{2!} + \cdots = \sum_{k\ge 0} \frac{B_k z^k}{k!}. \qquad (4)$$

Page 112 in (5) ——— 21 Mar 2021

$B_1 = -\frac{1}{2} \rightsquigarrow B_1 = \frac{1}{2}$

Page 112 replacement for line −3 ——— 21 Mar 2021

$$\frac{z}{1-e^{-z}} - \frac{z}{2} = \frac{z}{2}\,\frac{1+e^{-z}}{1-e^{-z}} = -\frac{z}{2}\,\frac{1+e^{z}}{1-e^{z}}$$

Page 112 append a footnote ——— 21 Mar 2021

* *Warning:* Many books, including earlier printings of *The Art of Computer Programming*, have used the formula $z/(e^z-1)$ instead of $z/(1-e^{-z})$ to define the Bernoulli numbers, thereby making $B_1 = -\frac{1}{2}$. But Peter Luschny published an online "Bernoulli Manifesto" in 2013, with compelling arguments that B_1 should be $+\frac{1}{2}$, as in the original works of Seki and Bernoulli.

Page 113 replacement for lines 1–12 ——— 21 Mar 2021

If we multiply both sides of the defining equation (4) by $1 - e^{-z}$, and equate coefficients of equal powers of z, we obtain the formula

$$\sum_k \binom{n}{k}(-1)^{n-k}B_k = B_n - \delta_{n1}. \qquad (7)$$

(See exercise 1.) We now define the *Bernoulli polynomial*

$$B_m(x) = \sum_k B_k\binom{m}{k}(x-1)^{m-k} = \sum_k \binom{m}{k}(-1)^k B_k x^{m-k}. \qquad (8)$$

If $m = 1$, then $B_1(x) = B_0x - B_1 = x - \frac{1}{2}$, corresponding to the polynomial used above in Eq. (3). If $m > 1$, we have $B_m(1) = B_m = B_m(0)$, by (6) and (7); in other words, $B_m(\{x\})$ has no discontinuities at integer points x.

The relevance of Bernoulli polynomials and Bernoulli numbers to our problem will soon be clear. We find by differentiating Eq. (8) that

$$\begin{aligned} B'_m(x) &= \sum_k \binom{m}{k}(m-k)(-1)^k B_k x^{m-k-1} \\ &= m\sum_k \binom{m-1}{k}(-1)^k B_k x^{m-1-k} \end{aligned}$$

Page 113 replacement for (10) ——— 21 Mar 2021

$$\begin{aligned} \sum_{k=1}^{n} f(k) &= \int_1^n f(x)\,dx + f(1) + \frac{B_1}{1!}\bigl(f(n)-f(1)\bigr) + \frac{B_2}{2!}\bigl(f'(n)-f'(1)\bigr) + \cdots \\ &\qquad + \frac{B_m}{m!}\bigl(f^{(m-1)}(n)-f^{(m-1)}(1)\bigr) + R_{mn} \\ &= \int_1^n f(x)\,dx + f(1) + \sum_{k=1}^{m}\frac{B_k}{k!}\bigl(f^{(k-1)}(n)-f^{(k-1)}(1)\bigr) + R_{mn}, \end{aligned} \qquad (10)$$

Page 114 line 7 ——— 26 Feb 2013

is less than $\leadsto$ is less in absolute value than

Page 114 replacement for (14) ——— 21 Mar 2021

$$H_n = \ln n + 1 + \sum_{k=1}^{m}\frac{B_k}{k}(-1)^{k-1}\Bigl(\frac{1}{n^k}-1\Bigr) + R_{mn}. \qquad (14)$$

Page 114 replacement for (15) ——— 21 Mar 2021

$$\gamma = \lim_{n\to\infty}(H_n - \ln n) = 1 + \sum_{k=1}^{m}\frac{B_k}{k}(-1)^k + \lim_{n\to\infty} R_{mn}. \qquad (15)$$

Page 114 lower half of page ——— 21 Mar 2021

lines −10 and −7: $H_{n-1} \leadsto H_n$
line −5: (adding $1/n$ to both sides) $\leadsto$

Page 115 replacement for (17) ——— 21 Mar 2021

$$\ln n! = n\ln n - n + 1 + \tfrac{1}{2}\ln n + \sum_{1<k\le m}\frac{B_k(-1)^k}{k(k-1)}\Bigl(\frac{1}{n^{k-1}}-1\Bigr) + R_{mn}. \qquad (17)$$

Page 115 in exercise 4 ——— 21 Mar 2021

line 3: $\sum_{0\le k<n} k^m \leadsto \sum_{k=0}^{n} k^m$
line −2: $0 \le k < n$ instead of $1 \le k < n \leadsto 0 \le k \le n$ instead of $1 \le k \le n$

Page 116 last line of exercise 6 ——— 21 Mar 2021

given in Section 1.2.5 $\leadsto$ given in Eq. 1.2.5–(15)

Page 116 line 2 of exercise 8 ——— 18 Feb 2012

$\binom{cn^2}{n}/c^n\binom{n^2}{n}$ ⤳ $\binom{cn^2}{n}/(c^n\binom{n^2}{n})$

Page 144 line 2 ——— 29 Nov 2011

to zero. ⤳ to zero, and the overflow toggle is cleared.

Page 145 line 5 ——— 28 Feb 2018

$X[i] \equiv$ `CONTENTS(X` $+\,i$`)` ⤳ `CONTENTS(X` $+\,i$`)` $\equiv X[i]$.

Page 151 line −11 ——— 29 Nov 2011

set to zero ⤳ *set to positive zero*

▸**Page 161** line 7 ——— 22 Mar 2018

$r_n(1/m_e) = r.$ ⤳ $r_n(1/m_e) = r.$ Stop if $r = 0$.

Page 173 line 3 after the table ——— 15 Dec 2013

table. ⤳ table, except that the unknown destination of e is represented there by ')' not '?'.

Page 196 line −4 ——— 28 Feb 2018

`ALF` ⤳ `ALF ␣␣␣␣␣`

▸**Page 197** lines 25–26 ——— 28 Feb 2018

rI1 possibly ⤳ rI1, rI4 possibly

▸**Page 221** line −13 (in 32nd to 40th printings only) ——— 08 May 2019

depicted in Fig. 24 ⤳ depicted in Fig. 27

▸**Page 229** line 11 ——— 06 Feb 2011

Mechanization ⤳ *Mechanisation*

Page 229 lines 25–29 ——— 19 Dec 2015

see *Proceedings* ... 87–90; B. E. Carpenter ... 78–79. ⤳ see B. E. Carpenter ... 78–79. See also H. D. Huskey's mention of "reversion storage" in *Proceedings* ... 87–90.

Page 242 line 2 ——— 28 Apr 2012

top, front ⤳ top, bottom, front

Page 261 lines 8 and 9 ——— 17 Aug 2021

a somewhat more natural alternative, and we will give ⤳
a more efficient alternative, and we will discuss

▸**Page 268** program line *71* ——— 30 Jun 2012

`QLINK(F)` ⤳ `QLINK[F]`

Page 268 replacement for lines −5 and −4 ——— 10 Nov 2021

In Section 7.2.1.2 we will study an efficient way to find *all* of the topological sorts of a given partial order relation.

Page 270 rewording of exercise 13 ——— 15 Mar 2018

13. [*HM48*] How many ways are there to arrange the 2^n subsets described in exercise 12 into topological order? (Try for a "sharp" asymptotic estimate as $n \to \infty$.)

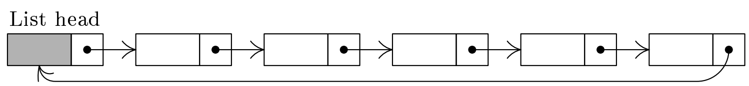
Page 273 replacement for exercise 30 ——— 17 Aug 2021

30. [*20*] (X. Xie, 2021.) Suppose that queues are represented almost as in (12), but with `R` pointing to a "blank" node just *after* the rear. (The queue is empty if and only if `F` = `R`; the contents of `INFO(R)` are irrelevant.) What insertion and deletion procedures should replace (14) and (17)?

Page 275 line 7 ——— 06 Sep 2012

list have $\rightsquigarrow$ list must have

Page 275 replacement for lines 10–13 ——— 16 Aug 2021



(4)

References to lists like (4) are usually made via the list head, which might occupy a fixed memory location. But a fixed list head leaves us without any pointer to the right end; so it forces us to sacrifice operation (b) stated above. Instead, we can retain (b) by letting the list head be mobile (see exercise 4).

Page 278 replacement for exercise 4 ——— 16 Aug 2021

▸**4.** [*20*] (X. Xie, 2021.) State insertion and deletion operations that give the effect of an output-restricted deque (in particular, of a stack and/or queue), using representation (4) with `PTR` pointing to the current list head.

▸**Page 286** table entry for time 4820, column `D3` ——— 03 Dec 2017

0 $\rightsquigarrow$ x

Page 303 in the line before (13) ——— 29 Aug 2011

transformation: $\rightsquigarrow$ transformation (see M. H. Doolittle, *Report of the Superintendent of the U.S. Coast and Geodetic Survey* (1878), 115–120):

Page 331 rating of exercise 14 ——— 15 Aug 2012

22 $\rightsquigarrow$ *20*

▸**Page 339** replacement for line 17 ——— 14 Mar 2011

$$D(y) = 3\bigl(1/(x+1)\bigr) - \bigl(-(a(2x))/(x^2)^2\bigr), \qquad (21)$$

Page 355 line 4 after (16) ——— 11 Nov 2021

Section 7.4.1 $\rightsquigarrow$ Section 7.4.1.1

Page 364 near the top ——— 20 Jan 2020

line 5: paths $(V, V_1, \ldots, V')$ and $(V, V_1', \ldots, V')$, $\rightsquigarrow$ paths $(V_0, V_1, \ldots, V_s)$ and $(V_0', V_1', \ldots, V_t')$, where $V_0 = V_0' = V$ and $V_s = V_t' = V'$,
line 7: path $(V_{k-1}, V_k', \ldots, V', \ldots, V_k)$ $\rightsquigarrow$
path $(V_{k-1}, V_k', \ldots, V_{t-1}', V', V_{s-1}, \ldots, V_k)$

▸**Page 374** line −2 ______ 22 Aug 2019

J. London Math. Soc. **21** (1947) $\rightsquigarrow$ *J. London Math. Soc.* (2) **21** (1946)

Page 385 replacement for line 5 ______ 12 Aug 2017

$$J = (D, U, \ , X) \qquad K = (\ , Y, R, L) \qquad N = (Y, \ , X, \ ) \qquad \epsilon = (\ , \ , \ , \ )$$

[Also, $B \rightsquigarrow \epsilon$ on lines 11–14, 17; omit B on lines 18 and 27.]

Page 407 lines 12–17 ______ 25 May 2017

Catalan numbers occur ... Catalan connection $\rightsquigarrow$

Catalan numbers occur in an enormous number of contexts: Richard Stanley's fascinating book *Catalan Numbers* (2015) features 214 different combinatorial interpretations, and also includes an appendix by Igor Pak that gives a definitive history of the subject. The most surprising of all these instances may well be the Catalan connection

Page 442 line 8 ______ 28 Mar 2021

uses one $\rightsquigarrow$ uses just one

Page 443 bottom line ______ 28 Mar 2021

as explained above. $\rightsquigarrow$ as explained above. (The user is supposed to know the value of k, from the current context, because a reserved block has no `KVAL` field.)

Page 465 new quotation (to follow the one by Lehmer) ______ 18 Aug 2013

I must explain, to begin with,
that all the Trees, in this system, grow head-downwards:
the Root is at the top, *and the Branches are* below.
If it be objected that the name "Tree" is a misnomer, my answer
is that I am only following the example of all writers on Genealogy.
A Genealogical *tree* always *grows* downwards:
then why may not a Logical *"Tree" do likewise?*
— LEWIS CARROLL, in *Symbolic Logic* (1896)

Page 466 new answer for exercise 6 ______ 28 Dec 2017

6. Trying Algorithm E with $n = 5$ and $m = 5q + (1, 2, 3, 4, 5)$, for any $q \geq 0$, we find that step E1 is executed (2, 3, 4, 3, 1) times, respectively. So the average is $2.6 = T_5$.

Page 466 append new copy at the end of answer 7 ______ 10 Aug 2014

For example, try $m = 5$ and $n = 1, 2, \ldots$: The average of 1, 2, 3, 2, 1, 3, 4, 5, 4, 2, 3, 4, 5, 4, 2, ... is 3.6.

Page 467 new line at end of answer 8 ______ 29 Apr 2013

Each iteration either decreases m or keeps m unchanged and decreases n.

Page 468 replacement for answer 2 ______ 19 Apr 2019

2. The formula for b involves division by zero when $n = 1$. Therefore the theorem has not been proved when $n + 1 = 2$.

Page 468 new answer ______ 13 Sep 2019

9. It's true when $n = 0$; also whenever it's known to be true for $n + 1$ or $n - 1$.

Page 469 new lines at end of answer 8 ——— 26 Feb 2013

This construction can't make $d_k = 9$ for all $k > l$, because that could happen only if $(n + d_1/10 + \cdots + d_l/10^l + 1/10^l)^m \le u$.

Page 469 line 1 of answer 9 ——— 01 Apr 2021

$= b^{pu}$, hence $\rightsquigarrow = b^{pu}$, using (6) twice. Hence

Page 470 line 2 of answer 11 ——— 17 Sep 2019

$10^x = 1.99999\ldots \rightsquigarrow 10^x$ begins with $1.99999\ldots$

Page 470 in step E1 of answer 28 ——— 28 Oct 2011

If $1 - \epsilon \ldots k \leftarrow 1$. $\rightsquigarrow$ Set $x \leftarrow 1 - \epsilon - x$, $y \leftarrow y_0$, and $k \leftarrow 1$, where $1 - \epsilon$ is the largest possible value of x, and y_0 is the nearest approximation to $b^{1-\epsilon}$.

Page 471 new answer 18 ——— 22 Dec 2019

18. $S'(j)$ sums over j with "$1 \le j < n$". $R'(i,j)$ sums over i with "n is a multiple of i and $i > j$".

Page 472 replacement for last line of answer 26 ——— 12 Jun 2020

The answer is $P_1 = (P_1P_2)^{1/2} = \bigl(\prod_{i=0}^n a_i\bigr)^{n+2}$, because $P_1 = \bigl(\prod_{i=0}^n \prod_{j=i}^n a_ia_j\bigr) = P_2$.

Page 472 new copy for end of answer 31 ——— 20 Oct 2011

Consequently we have $\bigl(\sum_{j=1}^n u_j\bigr)\bigl(\sum_{j=1}^n v_j\bigr) \le n\sum_{j=1}^n u_jv_j$ when $u_1 \le u_2 \le \cdots \le u_n$ and $v_1 \le v_2 \le \cdots \le v_n$, a result known as *Chebyshev*'s monotonic inequality. [See *Soobshch. mat. obshch. Khar'kovskom Univ.* **4**, 2 (1882), 93–98.]

Page 474 lines 3–5 of answer 43 ——— 26 Nov 2011

as in exercise 44 ... $(x_i - 1))$. $\rightsquigarrow$ by setting $x = 1$ in exercise 40 and obtaining $\prod_{k\ne i}(x_k - 1)/x_i \prod_{k\ne i}(x_k - x_i)$. After multiplying numerator and denominator by $x_i - 1$, we can sum on i by applying exercise 33 with $r = 0$ to the $n + 2$ numbers $\{0, 1, x_1, \ldots, x_n\}$.

Page 476 replacement for answer 4 ——— 22 Dec 2013

4. By part (f), $x \le \lceil x\rceil < x + 1$; hence $-x - 1 < -\lceil x\rceil \le -x$; use part (e).

Page 480 new answer ——— 03 Jun 2021

50. No: $\lfloor 3143/\pi\rfloor = 1000$. But $\lceil m\alpha\rceil = n$ does imply that $\lfloor n/\alpha\rfloor = m$, whenever $\alpha \ge 1$; and $\lfloor n/\alpha\rfloor = m$ does imply that $\lceil m\alpha\rceil = n$, whenever $0 < \alpha \le 1$.

▸**Page 484** lines 4 and 5 of answer 10 ——— 28 May 2018

terms $\rightsquigarrow$ factors (twice)

Page 485 replacement for answer 22 ——— 10 Sep 2013

22. Assume that $n > 0$. The kth term is $r/(r - tk)$ times

$$\frac{1}{n!}\binom{n}{k}\prod_{0\le j<k}(r - tk - j)\prod_{0\le j<n-k}(n - 1 - r + tk - j)$$
$$= \frac{(-1)^{k-1}}{n!}\binom{n}{k}\prod_{0\le j<k}(-r + tk + j)\prod_{k\le j<n}(-r + tk + j)$$

and the two products give a polynomial of degree $n - 1$ in k after division by $r - tk$. So the sum over k is zero by Eq. (34).

Page 485 lines 2 and 3 of answer 25 —— 07 Oct 2013

(Alternatively ... $x = 1$.) We have $\rightsquigarrow$ When w is sufficiently small, we have

▸**Page 486** last line of answer 30 —— 19 Sep 2021

the answer $\binom{n-1}{n-m}$ $\rightsquigarrow$ the answer $(-1)^{n-m}\binom{-m}{n-m}$

Page 490 line 5 of answer 62 —— 04 Oct 2021

factorial signs $\rightsquigarrow$ factorial factors

Page 491 line 2 of answer 64 —— 04 Oct 2021

$f(1,m) = \delta_{1m}$. If $n > 1$, $\rightsquigarrow$ $f(0,m) = \delta_{0m}$. If $n > 0$,

Page 491 line 2 of answer 66 —— 17 Apr 2015

$y > n-1 \rightsquigarrow y \geq n-1$

Page 491 replacement for last line of answer 67 —— 18 Feb 2012

$\binom{n}{k} \geq \bigl(\frac{(n-k+1)e}{k}\bigr)^k \frac{1}{ek}$, which is less memorable (but often sharper) than $\binom{n}{k} \geq \bigl(\frac{n}{k}\bigr)^k$.

Page 492 replacement for answer 16 —— 21 Jun 2016

16. $H_{2n} - \frac{1}{2}H_n$.

Page 493 new answer for section 1.2.7 —— 11 Feb 2013

25. $H_n^{(0,v)} = \sum_{k=1}^{n} H_n^{(v)}$ and $H_n^{(u,0)} = H_n^{(u-1)}$; so the identity generalizes (8). [See L. Euler, *Novi Comment. Acad. Sci. Pet.* **20** (1775), 140–186, §2.]

Page 495 last line of answer 34 —— 10 Jul 2018

Generalizations ... 7.1.3. $\rightsquigarrow$ For generalizations, see exercise 4.5.3–52, exercise 5.4.2–10, and Section 7.1.3.

▸**Page 498** line −2 of answer 21 —— 09 May 2021

Series (Dover, 1956) $\rightsquigarrow$ *Series*, second edition (Blackie, 1951)

▸**Page 499** last line of answer 23 —— 09 May 2021

27 (1965) $\rightsquigarrow$ **17** (1965)

Page 500 line −2 of answer 6 —— 21 Mar 2021

$H'(t) = e^t/(e^t-1) - 1/t \rightsquigarrow H'(t) = 1/(1-e^{-t}) - 1/t$

Page 503 replacement for answer 1 —— 21 Mar 2021

1. $(B_0+B_1z+B_2z^2/2!+\cdots)e^{-z} = (B_0+B_1z+B_2z^2/2!+\cdots) - z$; use Eq. 1.2.9–(11).

▸**Page 503** last line of answer 3 —— 05 Mar 2013

1937 $\rightsquigarrow$ 1927

Page 503 replacement for answer 4 —— 21 Mar 2021

4. $$S_m(n) = \frac{n^{m+1}}{1+m} + 0^m + \sum_{k=1}^{m} \frac{B_k}{k!} m^{\underline{k-1}} n^{m-k+1} = \frac{B_{m+1}(n+1) - B_{m+1}}{m+1} + 0^m.$$

Page 504 top half of page —— 21 Mar 2021

line 2: $\ln(n-1)! = (n-\frac{1}{2})\ln n \rightsquigarrow \ln n! = (n+\frac{1}{2})\ln n$
line 3: Now $\ln\Gamma_{n-1}(c) = c\ln(n-1) + \ln(n-1)! - \ln\bigl(c\ldots(c+n-1)\bigr)$; $\rightsquigarrow$ Now let $\Gamma_m(x) = m^x m!/x^{\overline{m+1}}$, so that $\ln\Gamma_n(c) = c\ln n + \ln n! - \ln\bigl(c(c+1)\ldots(c+n)\bigr)$;
line 2 of answer 7: $\sum_{k=1}^{n-1} \rightsquigarrow \sum_{k=1}^{n}$

Page 504 last line of answer 8 — 18 Feb 2012

$\binom{cn^2}{n}/c^n\binom{n^2}{n}$ ⟿ $\binom{cn^2}{n}/(c^n\binom{n^2}{n})$

Page 509 new answer 23 — 28 Feb 2018

23. First, by R. D. Dixon:

```
(3000) ENT1 4       DEC1 1
(3001) LDA  200     J1NN 3001
       SRA  0,1     SLAX 5
       SRAX 1       HLT  0   ▮
```

Second, by George Wahid, if rA ≥ 0:

```
(3000) ENT1 5:5          DEC1 1:1
(3001) SLA  1            J1P  3001
       ST1  3003(4:4)    HLT  0   ▮
(3003) ADD  200(*)
```

Page 514 replacement for line 5 of the program — 30 Apr 2013

```
3H     ENT2 9*8-8,1   Start at row 9.
```

▸**Page 514** replacement for lines 19 and 20 of the program — 10 Mar 2013

```
PHASE2 ENT3 9*8       At this point rA = min_j C(j)
3H     ENT2 0,3       Prepare to search a row.
```

Page 518 line 14 — 27 Aug 2014

Victorius of Aquitania ⟿ Victorius of Aquitaine

Page 540 lines 2–6 of answer 14 — 07 Jul 2021

When R is empty ... leaving the queue. ⟿ To insert at the rear, push onto S. To delete from the front, pop the top of R — except that if R is empty, first pop the elements of S onto R until S becomes empty. Each element is first pushed onto S, then popped from S and pushed to R, then popped from R when leaving the queue. (But if R and S are both empty, one might prefer to push directly onto R.)

▸**Page 545** line 8 of answer 3 — 23 Apr 2019

rA ← `INFO`(rI1) ⟿ rA ← `INFO`(rI3)

Page 546 last lines of answer 9 — 30 Jun 2014

plagues many automatic programming systems.) ⟿ plagued many automatic programming systems in the old days.)

Page 547 new answer 13 — 15 Mar 2018

13. Sha and Kleitman [*Discrete Math.* **63** (1987), 271–278] proved that the number is at most $\prod_{k=0}^{n}\binom{n}{k}^{\binom{n}{k}}$; Brightwell and Tetali [*Order* **20** (2003), 333–345] improved this upper bound to $\exp\bigl((2en2^n/r)^r\prod_{k=0}^{n}\binom{n}{k}!\bigr)$, where $r=(2^{n+1}-2)/(n+1)$. The obvious lower bound is $\prod_{k=0}^{n}\binom{n}{k}! = 2^{2^n(n+O(\log n))}$. For the initial values $\langle a_n\rangle = \langle 1,1,2,48,1680384,\ldots\rangle$, up to $a_7\approx 6.305\times 10^{137}$, see OEIS sequence A046873.

Page 551 replacement for answer 30 — 17 Aug 2021

30. To insert, set `P` ⇐ `AVAIL`, `INFO(R)` ← `Y`, `LINK(R)` ← `P`, `R` ← `P`, `LINK(P)` ← Λ. (We need not set `LINK(P)`; but that lack of discipline might confuse a garbage collection algorithm, as in exercise 6.) To delete, `UNDERFLOW` if `F` = `R`; otherwise set `P` ← `F`, `F` ← `LINK(P)`, `Y` ← `INFO(P)`, `AVAIL` ⇐ `P`.

Page 551 replacement for answer 4 — 17 Aug 2021

4. The text's operation (a) is unchanged; so is operation (c), but it's now preceded by "if `LINK(PTR)` = `PTR`, then `UNDERFLOW`." Operation (b) becomes "`INFO(PTR)` ← `Y`, `P` ⇐ `AVAIL`, `LINK(P)` ← `LINK(PTR)`, `LINK(PTR)` ← `P`, `PTR` ← `P`."

▶**Page 552** in answer 12 ——— 30 Jun 2012

line 1: $29p \leadsto 27p$
line 3: $\frac{3}{4} \leadsto 78\%$

Page 570 new sentence to follow line 1 of answer 30 ——— 08 Oct 2013

Thus `LOC(T)` = `HEAD`, and `HEAD$` is the first node of the binary tree in symmetric order.

Page 570 line −3 of answer 30 ——— 08 Oct 2013

the algorithm of exercise 21 $\leadsto$ Algorithm U in answer 21

Page 576 last line of answer 10 ——— 11 Nov 2021

Section 7.4.1 $\leadsto$ Section 7.4.1.1

Page 576 new answer(s) ——— 09 Sep 2014

14, 15. See Martin Ward and Hussain Zedan, "Provably correct derivation of algorithms using FermaT," *Formal Aspects of Computing* **26** (2014), 993–1031.

Page 578 last line of answer 6 ——— 11 Nov 2021

Section 7.4.1 $\leadsto$ Section 7.4.1.2

Page 579 line −8 of answer 8 ——— 18 Jul 2021

flows. We $\leadsto$ flows. (The unit flow in e_0 needn't be measured.) We

Page 587 replacement for top eight lines ——— 12 Aug 2017

types (αa, δNa, αd, δNa, αa, δNd, αd) on the main diagonal have the form

αa	βKQ	αb	βQP	αa	βK	αb
γJP	δNa	γR	δKQ	γJL	δNb	γP
αc	βDS	αd	βYQT	αc	βS	αd
γPQ	δJP	γXP	δNa	γRQ	δR	γR
αa	βKU	αb	βDP	αa	βK	αb
γJT	δNc	γS	δSD	γJS	δNd	γT
αc	βQS	αd	βDT	αc	βS	αd

Page 587 line 12 ——— 22 Jun 2019

For a similar $\leadsto$ (Six of the 92 types can actually be omitted; see exercise 7.2.2.1–121.) For a similar

Page 588 line 3 of answer 4 ——— 05 Jan 2020

at $z = \alpha$. $\leadsto$ at $z = \alpha$. (See exercise 7.2.2.2–348.)

Page 589 last line of answer 6 ——— 25 Feb 2018

140.] $\leadsto$ 140. Incidentally, there are g_{n-1} oriented trees with n *leaves* and in-degree 2 at nonleaves.]

Page 589 in answer 7 ——— 05 Jan 2020

$a \approx 2.483253536173$. $\leadsto$ and $a \approx 2.483253536173$ satisfies $F(1/a) = 0$.

Page 598 last line of answer 3 ———— 28 Dec 2017

two. ⇝ two. See exercises 7.2.1.6–26 through 7.2.1.6–34.

▸**Page 607** lines 1–9 ———— 24 Jun 2016

[In some printings, the "primes" disappeared from step names; for example, step C5′ was simply called step C5.]

▸**Page 611** line 5 ———— 25 Jul 2014

inner loop A3* ⇝ inner loop A2.5*

▸**Page 613** last line of answer 28 ———— 14 Apr 2014

`AVAIL[`k`]` ← `L`. ⇝ `AVAILF[`k`]` ← `L`.

Page 614 line −2 of answer 33 ———— 28 Dec 2017

Cohen and Nicolau ⇝ J. Cohen and A. Nicolau

Page 621 in Table 3 (the value of B_1) ———— 21 Mar 2021

$-1/2$ ⇝ $1/2$

Page 624 through page 626 ———— 07 Apr 2012

[replace the notation $(R \Longrightarrow x;\ y)$ by $(R?\ x{:}\ y)$ in eleven places]

Page 626 Bernoulli number entry ———— 21 Mar 2021

$(e^z - 1)$ ⇝ $(1 - e^{-z})$

Page 628 new entries for Appendix C ———— 17 Apr 2011

Program 1.2.10M, 145, 186.
Program 1.4.3.1M, 204–211, 530.
Program 2.1A, 236.
Program 2.1B, 535.
Program 2.3.1S, 325.

▸**Page 629** in Appendix C ———— 17 Apr 2011

Algorithm 2.4B′, 606. ⇝ Algorithm 2.4B′, 605. Algorithm 2.4B″, 606.
Algorithm 2.5G, 613. ⇝ Algorithm 2.5G, 613–614.

Page 630 and following ——— 01 Jan 2011

Miscellaneous changes to the existing index of Volume 1 are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.