

# Delay-Rate Tradeoff for Ergodic Interference Alignment in the Gaussian Case

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**Abstract**—In interference alignment, users sharing a wireless channel are each able to achieve data rates of up to half of the non-interfering channel capacity, no matter the number of users. In an ergodic setting, this is achieved by pairing complementary channel realizations in order to amplify signals and cancel interference. However, this scheme has the possibility for large delays in decoding message symbols. We show that delay can be mitigated by using outputs from potentially more than two channel realizations, although data rate may be reduced. We further demonstrate the tradeoff between rate and delay via a time-sharing strategy. Our analysis considers Gaussian channels; an extension to finite field channels is also possible.

## I. INTRODUCTION

The technique of interference alignment has expanded what is known about achievable rates for wireless interference channels. First proposed by Maddah-Ali *et al.* [1] and then applied to wireless interference channels by Cadambe and Jafar [2], interference alignment employs a transmission strategy that compensates for the interference channel between transmitters and receivers. At each receiver, the interference components can then be consolidated into a part of the channel that is orthogonal to the signal component. In fact, the interference is isolated to half of the received signal space, while the desired signal is located in the other half—leading to the statement that every receiver can have “half the cake.” This is a significant improvement over every receiver receiving only  $1/K$  of the cake, which is the case if standard orthogonalization techniques are used (where  $K$  is the number of transmitter-receiver pairs).

Interference alignment in an ergodic setting is studied in Nazer *et al.* [3], and provides the basis for our analysis. Using their Gaussian achievable scheme, we delve deeper into the associated decoding delays and consider how delays may be reduced, although at the cost of decreased rate. Even though the analysis in [3] additionally considers a scheme for finite field channels (also similar to the method in [4]), we defer to the reader the extension of our analysis to the finite field case.

Our approach for reducing delays is to consider interference alignment where alignment may require more than one additional instance of channel fading. In [3], interference is aligned by transmitting the same message symbol during

complementary channel realizations. In contrast, our approach will utilize multiple channel realizations (potentially more than two), which when summed together yield cancelled interference (and amplified signal). We call such a set of channel matrices an *alignment set*—which will be more formally defined later. Using multiple channel realizations to align interference has also been studied in [5] for different cases of receiver message requirements; however, we instead consider how to utilize these many channel realizations to reduce the delay of individual messages at each receiver. At first glance, it may seem that using alignment sets of larger sizes will only increase the delay; but if we allow alignment using alignment sets of multiple sizes simultaneously, then we can decrease the time required for a message symbol to be decoded.

We now give a simple example of an alignment set and show the concept of ergodic interference alignment.

*Example 1:* Consider a 3-user Gaussian interference channel with channel response given by  $\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{Z}$ , where  $\mathbf{X}$  denotes the transmitted symbols (with power constraint  $E[|X_k|^2] \leq P$  for each user  $k = 1, 2, 3$ ),  $\mathbf{H}$  is the channel matrix,  $\mathbf{Z}$  is independently and identically distributed zero-mean unit-variance additive white Gaussian noise, and  $\mathbf{Y}$  gives the received symbols. Suppose the following channel matrices occur at time steps  $t_0, t_1, t_2$ , and  $t_3$ , respectively:

$$\mathbf{H}^{(0)} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \quad \mathbf{H}^{(1)} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{H}^{(2)} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \quad \mathbf{H}^{(3)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

If the same [complex] vector  $\mathbf{X}$  is sent at all these times, then the sum of the non-noise terms is given by  $\sum_{i=0}^3 \mathbf{H}^{(i)} \mathbf{X} = 4[X_1, X_2, X_3]^T$  because  $\sum_{i=0}^3 \mathbf{H}^{(i)} = 4\mathbf{I}$ . By utilizing all four channel realizations together, the signals (diagonal entries) are amplified, while the interference terms (off-diagonal entries) are cancelled, so this collection of matrices is an alignment set. As long as a receiver knows when an alignment set occurs, then in order to decode his own message, he does not need to know the channel fades to the other receivers.

Inferring from [6] or [7], the astute reader may notice that in the example, the sum capacity when sending across each channel matrix separately is actually greater than the alignment rate—a capacity of  $4\log(1 + 3P)$  for separate coding, compared to a rate of  $3\log(1 + 4P)$  by using the indicated interference alignment scheme. However, when the

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number of transmitters (and receivers) exceeds the number of alignment channel realizations, then the rate benefits of using alignment sets start to become evident. Aligning across 4 channel realizations with  $K$  transmitter-receiver pairs, a rate of  $K \log(1+4P)$  is achievable, which can quickly eclipse the separate-coding sum capacity of  $4 \log(1+KP)$ . Moreover, as we will discuss, the benefit of using larger alignment sets is not in the rate, but rather in the reduction of decoding delay.

In the next section, we will formally describe the interference alignment setup, and define our notions of rate and delay. In Section III, we will take a brief look at the conventional ergodic interference alignment scheme, by considering the rate and delay inherent in aligning interference using complementary channel realizations. Section IV will give the main result of this work, which is the analysis of rate and delay when aligning interference by utilizing multiple channel realizations. We will also give a scheme for trading off the rate and the delay. We conclude in Section V.

## II. PRELIMINARIES

The setup is the same as the  $K$ -user interference channel of [3] and [5], where there are  $K$  transmitter-receiver pairs. The number of channel uses is  $n$ . For the  $k$ -th transmitter,  $k = 1, \dots, K$ , each message  $w_k$  is chosen independently and uniformly from the set  $\{1, 2, \dots, 2^{n\tilde{R}_k}\}$  for some  $\tilde{R}_k \geq 0$ . Only transmitter  $k$  knows message  $w_k$ . Let  $\mathcal{X}$  be the channel input and output alphabet. The message  $w_k$  is encoded into the  $n$  channel uses using the encoder  $\mathcal{E}_k : \{1, 2, \dots, 2^{n\tilde{R}_k}\} \rightarrow \mathcal{X}^n$ . The output of the encoding function is the transmitted symbol  $X_k(t) = [\mathcal{E}_k(w_k)]_t$  at time  $t$ , for  $t = 1, \dots, n$ .

The communication channel undergoes fast fading, so the channel fades change at every time step. At time  $t$ , the channel matrix  $\mathbf{H}(t)$  has complex entries  $[\mathbf{H}(t)]_{kl} = h_{kl}(t)$  for  $k, l = 1, \dots, K$ . In this model, all transmitters and receivers are given perfect knowledge of  $\mathbf{H}(t)$  for all times  $t$ . We call  $\mathcal{H}$  to be the set of all possible channel fading matrices.

The message symbol  $X_k(t)$  is transmitted at time  $t$ . We assume zero delay across the channel, so the channel output seen by receiver  $k$  at time  $t$  is the received symbol

$$Y_k(t) = \sum_{l=1}^K h_{kl}(t)X_l(t) + Z_k(t), \quad (1)$$

where  $Z_k(t)$  is an additive noise term. Each receiver  $k$  then decodes the received message symbols according to  $\mathcal{D}_k : \mathcal{X}^n \rightarrow \{1, 2, \dots, 2^{n\tilde{R}_k}\}$ , to produce an estimate  $\hat{w}_k$  of  $w_k$ .

*Definition 1:* The ergodic rate tuple  $(R_1, R_2, \dots, R_K)$  is achievable if for all  $\epsilon > 0$  and  $n$  large enough, there exist channel encoding and decoding functions  $\mathcal{E}_1, \dots, \mathcal{E}_K, \mathcal{D}_1, \dots, \mathcal{D}_K$  such that  $\tilde{R}_k > R_k - \epsilon$  for all  $k = 1, 2, \dots, K$ , and  $P\left(\bigcup_{k=1}^K \{\hat{w}_k \neq w_k\}\right) < \epsilon$ .

We assume a Gaussian channel with complex channel inputs and outputs, so  $\mathcal{X} = \mathbb{C}$ . Each transmitter  $k$  has power constraint

$$E[|X_k(t)|^2] \leq \text{SNR}_k,$$

where  $\text{SNR}_k \geq 0$  is the signal-to-noise ratio. The channel coefficients  $h_{kl}(t)$ ,  $k, l = 1, \dots, K$ , are independently and identically distributed both in space and time. We require also that  $h_{kl}$  be drawn from a distribution which is symmetric about zero, so  $P(h_{kl}) = P(-h_{kl})$ . The noise terms  $Z_k(t)$  are drawn independently and identically from a circularly-symmetric complex Gaussian distribution; thus,  $Z_k(t) \sim \mathcal{CN}(0, 1)$ .

### A. Channel Quantization

In this exposition, we consider quantized versions of the channel matrix. For some quantization parameter  $\gamma > 0$ , let  $Q_\gamma(h_{kl})$  be the closest point in  $(\mathbb{Z} + j\mathbb{Z})\gamma$  to  $h_{kl}$  in Euclidean distance. The  $\gamma$ -quantized version of the channel matrix  $\mathbf{H} \in \mathbb{C}^{K \times K}$  is given by the entries  $[\mathbf{H}_\gamma]_{kl} = Q_\gamma(h_{kl})$ .

Our scheme uses typical realizations of the channel matrices. For any  $\epsilon > 0$ , choose the maximum magnitude  $\tau > 0$  such that  $P(\bigcup_{k,l} \{|h_{kl}| > \tau\}) < \frac{\epsilon}{3}$ . Throw out all time indices with any channel coefficient magnitude larger than  $\tau$ . Let  $\gamma$  and  $\delta$  be small positive constants. Then choose  $n$  large enough so that the typical set of sequences  $A_\delta^n$  of channel matrices has probability  $P(A_\delta^n) \geq 1 - \frac{\epsilon}{3}$  (see [3] for details). Because this sequence of  $\gamma$ -quantized channel matrices is  $\delta$ -typical, the corresponding rate decrease is no more than a fraction of  $\delta$ .

In the remainder of this paper, we will only deal with the  $\gamma$ -quantized channel matrices  $\mathbf{H}_\gamma$ , so we drop the subscript  $\gamma$ ; all further occurrences of  $\mathbf{H}$  refer to the quantized channel realization  $\mathbf{H}_\gamma$ . We also redefine the channel alphabet  $\mathcal{H}$  to only include the typical set of quantized channel matrices, which has cardinality  $|\mathcal{H}| = (2\tau/\gamma)^{2K^2}$ .

### B. Aligning Interference

In the standard interference alignment approach, the interference is aligned by considering the channel matrix  $\mathbf{H}$  in tandem with its complementary matrix  $\mathbf{H}^c$ , where

$$\mathbf{H}^c = \begin{bmatrix} h_{11} & -h_{12} & \cdots & -h_{1K} \\ -h_{21} & h_{22} & \cdots & -h_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ -h_{K1} & -h_{K2} & \cdots & h_{KK} \end{bmatrix}.$$

That is,  $\mathbf{H}^c$  has entries  $h_{kl}$  for  $k = l$  and  $-h_{kl}$  for  $k \neq l$ .

For alignment using more channel realizations, we define the concept of an alignment set.

*Definition 2:* An alignment set of size  $m \in 2\mathbb{Z}^+$  is a collection of matrices  $\mathcal{A} = \{\mathbf{H}^{(0)}, \mathbf{H}^{(1)}, \dots, \mathbf{H}^{(m-1)}\}$  such that the diagonal entries (signal terms) are the same:

$$h_{kk}^{(0)} = h_{kk}^{(1)} = \dots = h_{kk}^{(m-1)} \quad (2)$$

for  $k = 1, \dots, K$ , and the sum of interference terms cancel:

$$\left| h_{kl}^{(0)} \right| = \left| h_{kl}^{(1)} \right| = \dots = \left| h_{kl}^{(m-1)} \right| \quad (3)$$

and

$$\left| \{h_{kl}^{(i)} = h_{kl}^{(0)} \mid i = 1, \dots, m-1\} \right| = \frac{m}{2} - 1 \quad (4)$$

$$\left| \{h_{kl}^{(i)} = -h_{kl}^{(0)} \mid i = 1, \dots, m-1\} \right| = \frac{m}{2} \quad (5)$$

for  $k = 1, \dots, K, l = 1, \dots, K, k \neq l$ . Within an alignment set, the sum of channel matrices,  $\mathbf{B} = \sum_{i=0}^{m-1} \mathbf{H}^{(i)}$ , will have entries  $b_{kk} = mh_{kk}^{(0)}$  and  $b_{kl} = 0$ , for  $k, l = 1, \dots, K, k \neq l$ . We denote  $\mathcal{A}_{\mathbf{H}}$  to be an alignment set of which  $\mathbf{H}$  is a member.

We have seen some examples of alignment sets already. Any channel realization  $\mathbf{H}$  and its complement  $\mathbf{H}^c$  together form an alignment set of size 2. Additionally, the set of matrices given in Example 1 is an alignment set of size 4.

Since channel transmission is instantaneous, the only delay considered is due to waiting for the appropriate channel realizations before a message symbol can be decoded.

*Definition 3:* The *average delay* of an ergodic interference alignment scheme is the expected number of time steps between the first instance a message symbol  $\mathbf{X}$  is sent and the time until  $\mathbf{X}$  is recovered at the receiver.

If  $\mathbf{X}(t_0)$  is sent at time  $t_0$  but can not be decoded until the appropriate interference alignment occurs at time  $t_1$ , then the delay is  $t_1 - t_0$ . Note that the delay does not consider the decoding of the entire message  $w_k$ —just the symbols transmitted at each individual time,  $X_k(t), k = 1, \dots, K$ .

### III. INTERFERENCE ALIGNMENT USING COMPLEMENTARY CHANNEL REALIZATION

The method of interference alignment via sending the same channel input vector when a complementary channel realization occurs is given in [3]. Call  $R_k^{(2)}$  the achievable rate for interference alignment using complements (*i.e.*, requiring two channel realizations before decoding each message symbol).

*Lemma 1 ([3, Theorem 3]):* An achievable rate tuple by aligning using complementary channel realizations is

$$R_k^{(2)} = \frac{1}{2} E[\log(1 + 2|h_{kk}|^2 \text{SNR}_k)]$$

for  $k = 1, \dots, K$ , where the expectation is over the distribution of channel fades  $h_{kk}$  drawn from the matrices in  $\mathcal{H}$ .

When a channel realization  $\mathbf{H}$  occurs, then the sent message symbol is decoded when the complementary channel realization  $\mathbf{H}^c$  occurs. Let  $d^{(2)}$  denote the average delay between channel realizations  $\mathbf{H}$  and  $\mathbf{H}^c$ .

*Lemma 2:* When all channel realizations are equally likely, the average delay incurred by interference alignment with complementary channel realizations is  $d^{(2)} = |\mathcal{H}|$ .

*Proof:* Each channel realization is equally likely at each time. The time until  $\mathbf{H}^c$  occurs is a geometric random variable with parameter  $P(\mathbf{H}^c) = 1/|\mathcal{H}|$ . The average delay is  $|\mathcal{H}|$ . ■

Note that the delay  $d^{(2)}$  can be quite large. Using our quantization scheme,  $d^{(2)} = |\mathcal{H}| = (2\tau/\gamma)^{2K^2}$ .

### IV. INTERFERENCE ALIGNMENT USING MULTIPLE CHANNEL REALIZATIONS

This section will focus on using alignment sets of sizes  $m = 2$  and  $m = 4$ . Extensions for larger alignment sets will be discussed in Section IV-C.

For ease of analysis, we assume that each channel realization  $\mathbf{H}$  is equally likely, although the ideas presented may be readily extended to the cases where the distribution of channel realizations is non-uniform. However, for this particular interference alignment scheme to work, all channel realizations within the same alignment set must be equiprobable: for an alignment set  $\mathcal{A}_{\mathbf{H}} = \{\mathbf{H}, \mathbf{H}^{(1)}, \mathbf{H}^{(2)}, \dots, \mathbf{H}^{(m-1)}\}$ , we require that  $P(\mathbf{H}) = P(\mathbf{H}^{(1)}) = P(\mathbf{H}^{(2)}) = P(\mathbf{H}^{(m-1)})$ . Fortunately, this holds since we assume that channel entries are drawn from distributions that are symmetric about zero.

#### A. First-to-Complete Alignment

We call the following scheme for achieving lower delay the *first-to-complete* scheme, which is essentially a coupon-collecting race between an alignment set of size 2 and an alignment set of size 4. For some channel realization  $\mathbf{H} \in \mathcal{H}$  (occurring at a time  $t_0$ )—since the entire future of channel realizations is known—we can collect the realizations occurring at future times  $t > t_0$ . Now we say that an alignment set  $\mathcal{A}_{\mathbf{H}}$  of size 4 has been *completed* once all matrices  $\tilde{\mathbf{H}} \in \mathcal{A}_{\mathbf{H}}$  have been realized. If  $\mathbf{H}^c$  occurs before  $\mathcal{A}_{\mathbf{H}}$  is completed, then pair up  $\mathbf{H}$  with that realization of  $\mathbf{H}^c$ . Otherwise, group together  $\mathbf{H}$  with the other members of the alignment set  $\mathcal{A}_{\mathbf{H}}$ .

We derive the achievable rate by separately finding the rates when decoding using alignment sets of different sizes, and then weighting these rates by the probabilities that a particular-sized set is completed before the other. From [3], if  $\mathbf{H}$  at time  $t_0$  is paired with  $\mathbf{H}^c$  at time  $t_1$ , then the same symbol vector  $\mathbf{X}(t_0)$  is transmitted at both times  $t_0$  and  $t_1$ . Since this is alignment with channel complements, the rate  $R_k = \frac{1}{2} E[\log(1 + 2|h_{kk}|^2 \text{SNR}_k)] - \epsilon$  is achievable with probability  $1 - \epsilon$ .

Now we find the rate when  $\mathbf{H}$  at time  $\hat{t}_0 = t_0$  is instead grouped with the members of its size-4 alignment set  $\mathcal{A}_{\mathbf{H}}$ . Assume that the channel realizations of the other members of the alignment set occur at times  $\hat{t}_1, \hat{t}_2$ , and  $\hat{t}_3$ , respectively. In the scheme, we send the same message symbol  $X_k(\hat{t}_0)$  at times  $\hat{t}_0, \hat{t}_1, \hat{t}_2$ , and  $\hat{t}_3$ . The channel outputs are

$$Y_k(t) = h_{kk}(t)X_k(\hat{t}_0) + \sum_{l \neq k} h_{kl}(t)X_l(\hat{t}_0) + Z_k(t) \quad (6)$$

for  $t = \hat{t}_0, \hat{t}_1, \hat{t}_2, \hat{t}_3$ . From the alignment set definition, we know  $h_{kk}(\hat{t}_0) = h_{kk}(\hat{t}_1) = h_{kk}(\hat{t}_2) = h_{kk}(\hat{t}_3)$  and  $h_{kl}(\hat{t}_0) + h_{kl}(\hat{t}_1) + h_{kl}(\hat{t}_2) + h_{kl}(\hat{t}_3) = 0$  for  $k = 1, \dots, K$  and  $l \neq k$ . Thus, the signal-to-interference-plus-noise ratio of the channel from  $X_k(\hat{t}_0)$  to  $Y_k(\hat{t}_0) + Y_k(\hat{t}_1) + Y_k(\hat{t}_2) + Y_k(\hat{t}_3)$  is at least

$$\frac{\text{SNR}_k((4|\Re(h_{kk})| - 2\gamma)^2 + (4|\Im(h_{kk})| - 2\gamma)^2)}{4 + (2\gamma)^2 \sum_{l \neq k} \text{SNR}_l}$$

Taking the channel quantization parameter  $\gamma \rightarrow 0$ , the SINR is  $4|h_{kk}|^2 \text{SNR}_k$ , which gives the rate (as  $\tau \rightarrow \infty$ ):

$$R_k = \frac{1}{4} E[\log(1 + 4|h_{kk}|^2 \text{SNR}_k)] - \frac{2\epsilon}{3}. \quad (7)$$

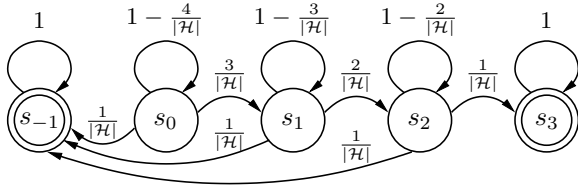


Fig. 1. Success runs Markov chain associated with first-to-complete alignment. States indicate progress towards completion of the alignment sets. Quantities above the arrows indicate transition probabilities.

Thus there exist  $\gamma$  and  $\tau$  such that we achieve  $R_k > \frac{1}{4}E[\log(1 + 4|h_{kk}|^2\text{SNR}_k)] - \epsilon$  with probability  $1 - \epsilon$  when aligning using an alignment set of size 4.<sup>1</sup>

Recall that  $\mathbf{H}$  at time  $t_0$  is only grouped with the channel realizations of the alignment set which completes first, so that the realizations corresponding to the other alignment sets are *not* associated with  $\mathbf{H}$  and can be used for some other transmissions. For example, if  $\mathbf{H}^c$  occurs between times  $\hat{t}_1$  and  $\hat{t}_2$  (i.e.,  $t_0 < \hat{t}_1 < t_1 < \hat{t}_2 < \hat{t}_3$ ), then since the transmitter knows the sequence of channel realizations in advance, it may avoid utilizing  $\mathbf{H}(\hat{t}_1)$  to send  $\mathbf{X}(t_0)$ , which would become a wasted transmission when  $\mathbf{H}^c$  occurs at time  $t_1$ . In this example, decoding is via channel complements, so  $\mathbf{X}(t_0)$  is sent during times  $t_0$  and  $t_1$ , but never during times  $\hat{t}_1$ ,  $\hat{t}_2$ , and  $\hat{t}_3$ .

We now determine the probability that the first-to-complete scheme decodes using the alignment set of size 4 rather than the alignment set of size 2. This can be computed by considering a Markov chain with the following states:

- $s_{-1}$ : Decode using  $\mathbf{H}$  and its complement,  $\mathbf{H}^c$
- $s_0$ : No matches yet to any alignment set
- $s_1$ : First match with size-4 alignment set
- $s_2$ : Second match with size-4 alignment set
- $s_3$ : Third match with size-4 alignment set, so decode using  $\mathcal{A}_H$

The Markov chain is shown in Figure 1. States  $s_{-1}$  and  $s_3$  are absorbing. Because this is a success runs Markov chain [8], its absorption probabilities and hitting times are known. The probability of decoding via the alignment set of size 4 is the probability of absorption at state  $s_3$  starting from state  $s_0$ , and is computed to be  $\beta_4 = 1/4$ . Note that  $\beta_4$  does not depend on the number of possible channel realizations,  $|\mathcal{H}|$ . This is intuitive since matrices not belonging to an alignment set do not affect the probability that one set completes before another.

*Lemma 3:* An achievable rate tuple for the first-to-complete scheme has rates (for all  $k = 1, \dots, K$ ):

$$R_k^{(2,4)} = \frac{3}{8}E[\log(1 + 2|h_{kk}|^2\text{SNR}_k)] + \frac{1}{16}E[\log(1 + 4|h_{kk}|^2\text{SNR}_k)].$$

*Proof:* Because decoding via the size-2 alignment set occurs  $1 - \beta_4$  of the time, and decoding via the size-4 alignment set occurs  $\beta_4$  of the time, an achievable rate is  $R_k^{(2,4)} = \frac{1}{2}(1 - \beta_4)E[\log(1 + 2|h_{kk}|^2\text{SNR}_k)] + \frac{1}{4}\beta_4E[\log(1 + 4|h_{kk}|^2\text{SNR}_k)]$ . Plugging in  $\beta_4 = 1/4$  gives the result. ■

<sup>1</sup>Higher rates may be possible by optimizing power allocations, for example via water-filling. Here we only consider rates achievable using equal-power allocations.

*Lemma 4:* For the first-to-complete scheme, the average decoding delay is  $d^{(2,4)} = (3/4)|\mathcal{H}| = (3/4)d^{(2)}$ .

*Proof:* The delay until either alignment set is completed is the mean hitting time until one of the corresponding absorption states is reached in the Markov chain of Figure 1. A simple computation for the hitting time yields  $d^{(2,4)} = (3/4)|\mathcal{H}|$ . ■

## B. Delay-Rate Tradeoff

Although the first-to-complete scheme achieves lower delay than interference alignment using only complements, it has the drawback of having lower rate. By using time-sharing, we can achieve any delay  $d$  such that  $(3/4)|\mathcal{H}| = d^{(2,4)} \leq d \leq d^{(2)} = |\mathcal{H}|$ , and every user  $k \in \{1, \dots, K\}$  will still have increased data rate over that of  $R_k^{(2,4)}$ .

In the time-sharing scheme, with probability  $1 - \alpha$  where  $0 \leq \alpha \leq 1$ , pair up  $\mathbf{H}$  with the first instance of  $\mathbf{H}^c$  which occurs later in time; this is alignment using only complements. With probability  $\alpha$ , however, perform the first-to-complete scheme: pair up  $\mathbf{H}$  with  $\mathbf{H}^c$  only if  $\mathbf{H}^c$  occurs before any alignment set of size 4 is completed; otherwise, group  $\mathbf{H}$  with the size-4 alignment set which completes first.

*Theorem 5:* The achievable rate when time-sharing with probability  $\alpha$  of using the first-to-complete scheme is

$$\begin{aligned} R_k(\alpha) &= (1 - \alpha)R_k^{(2)} + \alpha R_k^{(2,4)} \\ &= \frac{1}{2}\left(1 - \frac{\alpha}{4}\right)E[\log(1 + 2|h_{kk}|^2\text{SNR}_k)] \\ &\quad + \frac{\alpha}{16}E[\log(1 + 4|h_{kk}|^2\text{SNR}_k)]. \end{aligned}$$

*Proof:* Evident. ■

*Theorem 6:* The average delay when time-sharing is

$$d(\alpha) = (1 - \alpha)d^{(2)} + \alpha d^{(2,4)} = (1 - \alpha/4)|\mathcal{H}|.$$

*Proof:* Evident. ■

*Corollary 7:* The average delay, when time-sharing between the first-to-complete scheme (using alignment sets of both sizes 2 and 4) and channel-complement alignment, is lower than the average delay when using only complements.

*Proof:* By choosing any  $\alpha > 0$ , we get delay  $d(\alpha)$  strictly less than  $|\mathcal{H}| = d^{(2)}$ . ■

The reduced delay is an intuitive result since the first-to-complete scheme allows additional opportunities to align, without disallowing existing opportunities.

## C. Extension to Larger Alignment Sets

We now extend our analysis to more general collections of alignment sets. Consider a finite tuple of positive even numbers  $I = (m_1, m_2, \dots, m_{|I|})$ , possibly with repetitions. We generalize first-to-complete alignment by using non-overlapping alignment sets with sizes dictated by the entries of  $I$ . As soon as all members of any particular alignment set have been seen, we say that that alignment set has been completed; we transmit and decode using the particular alignment set. As an example, the first-to-complete alignment scheme given in the first part of this section corresponds to  $I = (2, 4)$ . For the case of a general tuple  $I$ , the process is identical to the multiple subset coupon collecting problem of Chang and Ross [9], in which coupons are repeatedly drawn

TABLE I  
ABSORPTION PROBABILITIES AND DELAYS<sup>†</sup>

Set sizes $I$	Absorption probability			Delay
	$\beta_{m_1}^I$	$\beta_{m_2}^I$	$\beta_{m_3}^I$	$d^I$
(2, 4)	0.75	0.25		$0.75 \mathcal{H} $
(2, 6)	0.8333	0.1667		$0.8333 \mathcal{H} $
(2, 4, 4)	0.6429	0.1786	0.1786	$0.6429 \mathcal{H} $
(2, 4, 6)	0.6944	0.2083	0.0972	$0.6944 \mathcal{H} $
(4, 4)	0.5	0.5		$1.2167 \mathcal{H} $
(4, 6)	0.625	0.375		$1.3988 \mathcal{H} $
(4, 8)	0.7	0.3		$1.4972 \mathcal{H} $
(4, 4, 4)	0.3333	0.3333	0.3333	$0.9790 \mathcal{H} $
(6, 10)	0.6429	0.3571		$1.8607 \mathcal{H} $

<sup>†</sup> For values to be valid,  $|\mathcal{H}| \geq 1 + \sum_{i=1}^{|I|} (m_i - 1)$  must hold.

with replacement until any one of several preordained subsets of coupons have been collected.

To compute the achievable rates  $(R_1^I, R_2^I, \dots, R_K^I)$  and delay  $d^I$  associated with running first-to-complete alignment among  $I$ -sized alignment sets, we construct the associated Markov chain. The state vector  $\mathbf{s} = (s_1, s_2, \dots, s_{|I|})$  is defined so that element  $s_i$  counts how many members of the  $i$ -th alignment set have already occurred, excluding the initial matrix  $\mathbf{H}$ . Initially, the Markov chain is at state  $\mathbf{s} = \mathbf{0}$ , since no alignment set member aside from  $\mathbf{H}$  has yet been realized. At each time  $t$ , if  $\mathbf{H}(t)$  is a member of the  $\hat{i}$ -th alignment set and has not yet been realized, then increment  $s_{\hat{i}} := s_{\hat{i}} + 1$ . When  $s_{\hat{i}} = m_{\hat{i}} - 1$  for some  $\hat{i}$ , this means that the  $\hat{i}$ -th alignment set (of size  $m_{\hat{i}}$ ) has been completed. The Markov chain enters an absorbing state, and the receiver decodes. Let  $V$  denote the set of absorbing states. The state transition probabilities are

$$P_{\mathbf{s}, \mathbf{s}'} = \begin{cases} \frac{m_{\hat{i}} - 1 - s_{\hat{i}}}{|\mathcal{H}|} & s'_{\hat{i}} = s_{\hat{i}} + 1 \text{ for some } \hat{i}, \dots \\ & s'_i = s_i \text{ for all } i \neq \hat{i}, \mathbf{s} \notin V \\ 1 - \sum_i \frac{m_i - 1 - s_i}{|\mathcal{H}|} & \mathbf{s}' = \mathbf{s}, \mathbf{s} \notin V \\ 1 & \mathbf{s}' = \mathbf{s}, \mathbf{s} \in V \text{ (absorption)} \\ 0 & \text{otherwise} \end{cases}.$$

Let  $\beta_m^I$  be the probability that the first completed alignment set is the alignment set of size  $m \in I$ . Equivalently,  $\beta_m^I$  is the probability that the Markov chain reaches the absorption state corresponding to the completion of a specific size- $m$  alignment set. These absorption probabilities can be computed via matrix inversion (see the Appendix or Taylor and Karlin [8] for more details). Table I gives example values for  $\beta_m^I$ .

Following a similar argument as in Lemma 3, the rate for receiver  $k \in \{1, \dots, K\}$  by using a first-to-complete scheme with specific alignment sets of sizes drawn from  $I$  is

$$R_k^I = \sum_{m \in I} \frac{1}{m} \beta_m^I E[\log(1 + m|h_{kk}|^2 \text{SNR}_k)].$$

We now incorporate time-sharing and describe the delay-rate tradeoff. Let  $\mathcal{I}$  be a finite collection of these tuples  $I$ ; that is,  $\mathcal{I} \subseteq \{I = (m_1, \dots, m_{|I|}) \mid m_i \in 2\mathbb{Z}^+\}$ . We can do time-sharing between first-to-complete schemes, with sizes drawn from  $I \in \mathcal{I}$ , according to the vector  $\boldsymbol{\alpha} =$

$(\alpha_{I_1}, \alpha_{I_2}, \dots, \alpha_{I_{|\mathcal{I}|}})$  where  $\sum_{I \in \mathcal{I}} \alpha_I = 1$  and  $\alpha_I \geq 0$  for all  $I \in \mathcal{I}$ . The rate will be

$$R_k(\boldsymbol{\alpha}) = \sum_{I \in \mathcal{I}} \alpha_I R_k^I. \quad (8)$$

Alternatively, to be explicit about the rates due to alignment sets of particular sizes, the rate can also be written as

$$R_k(\boldsymbol{\alpha}) = \sum_{m \in 2\mathbb{Z}^+} \left( \sum_{I \in \mathcal{I} : m \in I} \alpha_I \beta_m^I \right) \frac{1}{m} E[\log(1 + m|h_{kk}|^2 \text{SNR}_k)].$$

The average delay using alignment sets of sizes  $I = (m_1, m_2, \dots, m_{|I|})$  is equal to the mean absorption time for the Markov chain. From [9], by using Poisson embedding, this delay can be computed as<sup>2</sup>

$$d^I = |\mathcal{H}| \int_0^1 \frac{1}{1-u} \prod_{i=1}^{|I|} (1 - u^{m_i-1}) du. \quad (9)$$

Table I gives average delays for some representative collections of alignment sets. Then the delay using time-sharing is

$$d(\boldsymbol{\alpha}) = \sum_{I \in \mathcal{I}} \alpha_I d^I, \quad (10)$$

which is linear in the number of possible channel realizations,  $|\mathcal{H}|$ .

From Table I, we can make an observation regarding the computed absorption probabilities and associated delays. When the first alignment set has size 2, notice that  $d^I = \beta_2^I |\mathcal{H}|$ . This holds for any tuple  $I$  which contains an alignment set of size 2 (see Appendix).

#### D. Further Considerations

In this analysis, we only consider alignment sets that do not share any common matrices. However, as the number of allowable sizes,  $|I|$ , grows larger, this condition will become harder to fulfill since there will be greater potential for collisions. Finding tuples of alignment sets such that there are no overlapping channels is an avenue for future work. One thing to note is that because only  $2^{K(K-1)}$  matrices satisfy  $h_{kk}^{(i)} = h_{kk}$  and  $|h_{kl}^{(i)}| = |h_{kl}|$  for  $k = 1, \dots, K$  and

<sup>2</sup>This evaluates to an inclusion-exclusion sum of harmonic numbers  $H_n$ :

$$d^I = |\mathcal{H}| \left[ \sum_{U \subseteq I, U \neq \emptyset} (-1)^{1-|U|} H_{(-|U| + \sum_{m \in U} m)} \right].$$

The delay can also be expressed analytically using the digamma function  $\Psi$ , giving

$$\begin{aligned} d^I &= |\mathcal{H}| \sum_{U \subseteq I} (-1)^{1-|U|} \Psi \left( 1 - |U| + \sum_{m \in U} m \right) \\ &= |\mathcal{H}| \left[ \gamma + \sum_{m_1 \in I} \Psi(m_1) - \sum_{m_1, m_2 \in I} \Psi(-1 + m_1 + m_2) \right. \\ &\quad \left. + \sum_{m_1, m_2, m_3 \in I} \Psi(-2 + m_1 + m_2 + m_3) - \dots \right], \end{aligned}$$

where  $\gamma$  is the Euler-Mascheroni constant. Also, from [9], we can find the variance of this delay, as well as the average delay when alignment sets overlap.

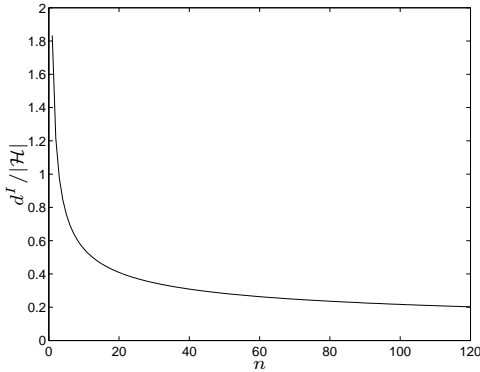


Fig. 2. Plot showing the decrease of the delay linear scaling factor, for multiple disjoint alignment sets of size 4. The number of alignment sets of size 4 is  $n$ . Thus each point represents the delay associated with the tuple  $I = (4, 4, 4, \dots)$ , where the tuple has  $n$  elements.

$l \neq k$ , an alignment set of size  $m = 2^{K(K-1)}$  would consist of all possible channel matrices which might align with  $\mathbf{H}$ , and so necessarily must collide with any other alignment set.

A related issue is that of allowing decoding using *all* alignment sets of a particular size  $m$ , of which there are  $\binom{m-1}{m/2}^{K(K-1)}$  such alignment sets. For example, a system could choose to perform first-to-complete alignment among *any* alignment set of sizes 2 and 4. Because non-intersection between different alignment sets may no longer be guaranteed, the analysis will be more complicated.

From Table I, we can start to notice the potential for delay reduction via using multiple alignment sets of the same size. Although the delay will still scale linearly in  $|\mathcal{H}|$ , it is possible to significantly reduce the delay below  $d^{(2)} = |\mathcal{H}|$ . As an example, from Figure 2 we can observe the behavior of the linear scaling factor, in the case of allowing alignment using more and more size-4 alignment sets.<sup>3</sup> Thus a deeper consideration of alignment with multiple same-size alignment sets may be a fruitful area for further inquiry.

There are myriad other ways in which alignment may occur; *i.e.*, there is more than one way to align channel matrices. Definition 2 gives one set of sufficient conditions for channel realizations to align, in order to keep the analysis tractable—and the benefits which arise by considering larger alignment sets are already evident. An obvious extension to this would be to consider alignment sets in which arbitrary linear combinations add up to multiples of the identity, and to only consider alignment among subsets of users. Subsequent work by [10] takes a step in this direction.

The moral of this story, however, is that delay can always be reduced by allowing alignment using a greater number of possible choices of alignment sets. The data rate may decrease correspondingly, so the tradeoff needs to be appropriately chosen according to the needs of the communication system.

<sup>3</sup>Of course, the trend shown in the Figure 2 only holds for scenarios where the number of users  $K$  is large enough that there exists enough distinct alignment sets of size 4 for alignment.

## V. CONCLUSION

In our analysis, we have not considered the delays between when a message symbol is available and when it is first transmitted. We have only defined delay as the time between when the symbol is first transmitted and when it is able to be recovered by the receiver. We believe this is a reasonable metric of delay, as long as message symbols are not all generated at one time. However, an analysis using queueing theory may be necessary to verify this claim.

In this work, we have proposed an interference alignment scheme which reduces delay, although with potentially decreased data rate. Delay is mitigated by allowing more ways to align interference—through the utilization of larger alignment sets. We have also introduced a scheme to trade off the delay and rate. In the end, even though the rate may be reduced, we can still say, in the parlance of interference aligners, that each person gets  $\kappa$  of the cake, where  $1/K \leq \kappa \leq 1/2$ —so our scheme can still be an improvement over non-aligning channel-sharing strategies in terms of data rate.

## APPENDIX

### MARKOV CHAIN ANALYSIS

We provide more details on computing the absorption probabilities and hitting times from the Markov chain constructions of Section IV, using techniques from [8]. Assume there are a total of  $n$  states in the Markov chain, with  $k$  transient states and  $n - k$  absorbing states. In the rest of the appendix, let  $e_i$  denote a vector consisting of all 0's except for a 1 in the  $i$ -th position (*i.e.*,  $e_i$  is the canonical basis vector in the  $i$ -th direction). We let state  $i = 0$  be the initial state of the Markov chain—with no alignment sets completed—so  $e_0$  is the initial probability distribution. Also, let  $\mathbf{1}$  be the all-ones vector (of appropriate length).

Consider the  $n \times n$  probability transition matrix  $\mathbf{P}$ , with the  $P_{ij}$  entry denoting the probability of transitioning from state  $i$  to state  $j$ . Without loss of generality, we may reorder the states so that the transient states are indexed first, and then followed by the absorbing states. Equivalently, we permute the rows and columns of  $\mathbf{P}$  to have the block upper-triangular form  $\mathbf{P} = \begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$ , where the block  $\mathbf{Q}$  (of size  $k \times k$ ) corresponds to transition probabilities between transient states and the block  $\mathbf{R}$  (of size  $k \times (n - k)$ ) corresponds to transition probabilities from transient states to absorbing states. (The lower-right block is the identity matrix since an absorbing state can only transition to itself, and obviously the lower-left block is all zeros since absorbing states can not transition to transient states.) As an example, if we consider the Markov chain of Figure 1 with re-ordered state vector  $\mathbf{s} = (s_0, s_1, s_2, s_3, s_{-1})$ , then the permuted probability transition matrix is

$$\mathbf{P} = \begin{bmatrix} 1 - \frac{4}{|\mathcal{H}|} & \frac{3}{|\mathcal{H}|} & 0 & 0 & \frac{1}{|\mathcal{H}|} \\ 0 & 1 - \frac{3}{|\mathcal{H}|} & \frac{2}{|\mathcal{H}|} & 0 & \frac{1}{|\mathcal{H}|} \\ 0 & 0 & 1 - \frac{2}{|\mathcal{H}|} & \frac{1}{|\mathcal{H}|} & \frac{1}{|\mathcal{H}|} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which evidently has the appropriate structure.

Expressions for the absorption probabilities and hitting times can be derived using the various blocks of the probability transition matrix.

*Lemma 8:* Define the length- $(n - k)$  absorption probability vector  $\beta$ , where the  $\beta_j$  entry is the probability of becoming absorbed in state  $j$ . Then

$$\beta = (e_0^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{R})^T.$$

*Proof:* We consider the  $k \times (n - k)$  transient-to-absorbing matrix  $\mathbf{U}$ , where the  $U_{ij}$  entry denotes the probability of starting in transient state  $i$  and ultimately becoming absorbed in absorbing state  $j$ . By first-step analysis,  $\mathbf{U}$  satisfies the recursion  $\mathbf{U} = \mathbf{Q}\mathbf{U} + \mathbf{R}$ , so  $\mathbf{U} = (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{R}$ . Because  $\mathbf{Q}$  represents the probabilities of transitioning between transient states,  $(\mathbf{I} - \mathbf{Q})^{-1}$  is the fundamental matrix and is well-defined. Then  $\beta_j$  is the probability of starting in state 0 and eventually becoming absorbed in state  $j$ , so  $\beta = (e_0^T \mathbf{U})^T = (e_0^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{R})^T$ . ■

*Lemma 9:* The hitting time (*i.e.*, the time until absorption in any absorption state) is given by

$$d = e_0^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}.$$

*Proof:* Let  $\mathbf{D}$  be the length- $k$  vector where the  $D_i$  entry is the hitting time when starting in transient state  $i$ . Then first-step analysis gives the recursion  $\mathbf{D} = \mathbf{Q}\mathbf{D} + \mathbf{1}$ , so  $\mathbf{D} = (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}$ . The overall hitting time is then  $d = e_0^T \mathbf{D} = e_0^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}$ . ■

Suppose one employs the first-to-complete alignment scheme with alignment sets of sizes  $I = (m_1, m_2, \dots, m_{|I|})$ , and where the first alignment set has size  $m_1 = 2$ . Here we prove that the mean time to absorption of the Markov chain is equal to the number of possible channel fading matrices multiplied by the probability of completion using the set of size 2.

*Theorem 10:* If  $2 \in I$ , then  $d^I = \beta_2^I |\mathcal{H}|$ .

*Proof:* Let  $\hat{j}$  be the state associated with the realization of the channel complement (*i.e.*, the state associated with completing the size-2 alignment set). We assume that each channel realization is equally likely with probability  $1/|\mathcal{H}|$ , so the probability of transitioning into state  $\hat{j}$  is  $1/|\mathcal{H}|$  starting

from any [transient] state. From Lemma 8 and since the  $\hat{j}$ -th column of  $\mathbf{R}$  is  $(1/|\mathcal{H}|)\mathbf{1}$ , we see that  $\beta_2^I = \beta_{\hat{j}} = (1/|\mathcal{H}|)e_0^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}$ . Since  $d^I = e_0^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}$ , the result follows. ■

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