Maximum Algebraic Connectivity Augmentation is NP-Hard

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Abstract

The algebraic connectivity of a graph, which is the second-smallest eigenvalue of the Laplacian of the graph, is a measure of connectivity. We show that the problem of adding a specified number of edges to an input graph to maximize the algebraic connectivity of the augmented graph is NP-hard.
Keywords: Algebraic connectivity; Computational complexity

1 Introduction

Two natural measures of the connectivity of a graph are vertex connectivity and edge connectivity. An alternative measure is algebraic connectivity, introduced by Fiedler [1], which is defined as an eigenvalue of a certain matrix associated with a graph. In some applications, algebraic connectivity can be a more useful measure of the connectivity of a graph than vertex or edge connectivity. For example, all trees have the same vertex and edge connectivity (one), whereas the algebraic connectivity of a star is higher than that of a path. It is well known that graphs with large algebraic connectivity have many applications [4].

In this work, we study an optimization problem involving the algebraic connectivity of graphs. Given an input graph, our goal is to add a given number of edges to the graph to maximize the algebraic connectivity of the augmented graph. This problem models, for example, the task of building infrastructure to support routing between nodes in a network. It was previously studied by Maas [7], as it arises in a description of the flow of a liquid through a system of communicating pipes, and Ghosh and Boyd [3]. These papers present a greedy heuristic for adding edges to the graph, but leave the computational complexity of the problem open. The complex objective function, being an eigenvalue of a matrix, suggests that the problem is likely to be difficult. We confirm this intuition here by proving that the problem is NP-hard.

The remainder of the section introduces the optimization problem and our result. Let $G = (V, E)$ be a graph on the vertex set $V = \{1, \ldots, n\}$. Throughout this paper, we assume that all graphs are undirected and simple, but we do not assume that graphs are necessarily connected. The degree of a vertex $v \in V$ is denoted by $\deg(v)$. For a graph $G = (V, E)$, let $G^c = (V, E^c)$ denote the complementary graph of G, where $E^c = \{(u, v) \mid u, v \in V, u \neq v, (u, v) \notin E\}$.

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Definition 1. The *Laplacian* $L(G)$ of G is the $n \times n$ matrix L defined by

$$L_{ij} = \begin{cases} \deg(i) & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } (i,j) \in E, \\ 0 & \text{if } i \neq j \text{ and } (i,j) \notin E. \end{cases}$$

For a vector $x \in \mathbf{R}^n$, the quadratic form $x^T L(G) x$ is

$$x^T L(G) x = \sum_{(i,j) \in E} (x_i - x_j)^2,$$

where the sum is to be interpreted as zero when the edge set E is empty. As a result, $L(G)$ is a symmetric positive semidefinite matrix with non-negative eigenvalues, which we denote by $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$. We also write $\lambda_{\max}(G)$ to denote the largest eigenvalue of $L(G)$, $\lambda_n(G)$. The n-dimensional vector containing a one in each component is an eigenvector of $L(G)$ corresponding to the eigenvalue zero, so $\lambda_1(G) = 0$.

Definition 2. The *algebraic connectivity* of a graph G is the second-smallest eigenvalue of $L(G)$, $\lambda_2(G)$.

We show that the following decision version of the maximum algebraic connectivity augmentation optimization problem is NP-hard.

Definition 3. The *maximum algebraic connectivity augmentation problem* is defined as follows.
Instance: An undirected graph $G = (V, E)$, a non-negative integer k, and a non-negative threshold t.
Question: Is there a subset $A \subseteq E^c$ of size $|A| \leq k$ such that the graph $H = (V, E \cup A)$ satisfies $\lambda_2(H) \geq t$?

Note that this problem is also the decision version of the following optimization problem: Given an input graph and a threshold, find a set of edges of minimum size to add to the graph so that the algebraic connectivity of the augmented graph is at least the threshold. As such, our hardness result applies to both this optimization problem and the problem of adding a specified number of edges to a graph to maximize the algebraic connectivity of the augmented graph.

A verifier can determine whether $\lambda_2(H) \geq t$ for an augmented graph H and an input threshold t by testing whether the matrix $L(H) - t(I - U)$ is positive semidefinite, where U is an $n \times n$ matrix with each entry equal to $1/n$. Since the test can be performed using Gaussian elimination in polynomial time [6], the maximum algebraic connectivity augmentation problem is in NP. We show that it is NP-complete.

Theorem 1. *The maximum algebraic connectivity augmentation problem is NP-hard.*

2 Proof of Theorem 1

We first present several properties of algebraic connectivity and the largest eigenvalue of the Laplacian that will be useful in the proof.

Property 2 (Rayleigh-Ritz [5]). *For any graph $G = (V, E)$ with $|V| = n$ vertices,*

$$\lambda_{\max}(G) = \max_{x \in \mathbf{R}^n, x \neq \vec{0}} \frac{x^T L(G) x}{x^T x}.$$

The following properties are developed in the seminal work of Fiedler [1].

Property 3. *For any graph $G = (V, E)$ with $|V| = n$ vertices, $\lambda_2(G) = n - \lambda_{\max}(G^c)$.*

Property 4. *For a graph $G = (V, E)$, let $C_1, \ldots, C_p$ denote the connected components of G. Then $\lambda_{\max}(G) = \max_{i=1,\ldots,p} \lambda_{\max}(C_i)$.*

Property 5. *Let K_n denote the complete graph on $n > 1$ vertices. Then $\lambda_{\max}(K_n) = n$.*

Property 6. *For any graph $G = (V, E)$ that is not complete, $\lambda_2(G) \leq \min_{v \in V} \deg(v)$.*

Property 6 follows from the work of Fiedler [1]. It is also a consequence of Property 3 and the following property, which we prove here and use later in the NP-hardness proof.

Property 7. *Let $G = (V, E)$ be a graph containing a vertex $v \in V$ with degree $\deg(v) = d > 0$, and let $S \subseteq V$ be the set of vertices in the connected component containing v. Then $\lambda_{\max}(G) \geq d + 1$. In addition, if $|S| > d + 1$, then $\lambda_{\max}(G) > d + 1$.*

Proof. Define a vector x on the vertices in V as

$$x_u = \begin{cases} d & \text{if } u = v, \\ -1 & \text{if } u \neq v \text{ and } (u, v) \in E, \\ 0 & \text{if } u \neq v \text{ and } (u, v) \notin E. \end{cases}$$

For the vector x, $x^T x = d^2 + d(-1)^2 = d(d + 1)$. The quadratic form $x^T L(G) x$ has a term $(x_u - x_v)^2$ for each of the d edges of the form (u, v) incident on v in G. Each of these terms contributes $(d - (-1))^2 = (d + 1)^2$ to the value of the quadratic form, so $x^T L(G) x \geq d(d + 1)^2$. By Property 2,

$$\lambda_{\max}(G) \geq \frac{x^T L(G) x}{x^T x} \geq \frac{d(d + 1)^2}{d(d + 1)} = d + 1. \tag{1}$$

If the set S has $|S| > d + 1$ vertices, then, because the degree of v is d, the connected component containing v also contains a vertex that is not incident on v, but from which there is a path to v. As a result, there must be an edge $(i, j) \in E$ such that $x_i = -1$ and $x_j = 0$. This edge contributes $(-1 - 0)^2 = 1$ to the value of the quadratic form, so $x^T L(G) x \geq d(d + 1)^2 + 1 > d(d + 1)^2$. The inequality in (1) now becomes strict, and thus $\lambda_{\max}(G) > d + 1$. $\square$

The proof of NP-hardness of the maximum algebraic connectivity augmentation problem is by reduction from 3-colorability [2]. Given a graph $G = (V, E)$ with $|V| = n > 1$ vertices and $|E| = m$ edges, we construct a graph $G' = (V', E')$ that is three disjoint copies of G. Formally, the vertex set V' and the edge set E' are defined as follows:

$$V' = \bigcup_{v \in V} \{v_0, v_1, v_2\};$$

$$E' = \bigcup_{(u,v) \in E} \{(u_0, v_0), (u_1, v_1), (u_2, v_2)\}.$$

Note that the graph G' contains $3n$ vertices and $3m$ edges. The instance of the maximum algebraic connectivity augmentation problem output by the reduction consists of the graph G', together with $k = 3n^2 - 3m$ (so the augmented graph H will contain at most $3n^2$ edges), and $t = 2n$.

Let $K_{n,n,n}$ denote the complete 3-partite graph with n vertices in each of the three partitions. The correctness of the reduction follows from these two lemmas.

Lemma 8. *There exists a subset $A \subseteq (E')^c$ of size $|A| \leq k$ such that $H = (V', E' \cup A)$ is (isomorphic to) $K_{n,n,n}$ if and only if G is 3-colorable.*

Proof. If G is 3-colorable, there exists a function $f : V \to \{0, 1, 2\}$ such that $f(u) \neq f(v)$ whenever $(u, v) \in E$. Define a function $g : V' \to \{0, 1, 2\}$ by, for all $v \in V$ and $i = 0, 1, 2$, $g(v_i) = (f(v) + i) \bmod 3$. Then, g is a 3-coloring of G' in which exactly n vertices are assigned each of the three colors. By adding the $3n^2 - 3m$ edges not present in G' that connect vertices assigned different colors by g, we obtain the complete 3-partite graph $K_{n,n,n}$.

Suppose that there is a subset $A \subseteq (E')^c$ of size $|A| \leq k$ such that $H = (V', E' \cup A)$ is isomorphic to $K_{n,n,n}$. Since $K_{n,n,n}$ is 3-colorable, there exists a 3-coloring $g : V' \to \{0, 1, 2\}$ of H. Define a function $f : V \to \{0, 1, 2\}$ by $f(v) = g(v_0)$ for all $v \in V$. For any $(u, v) \in E$, the edge (u_0, v_0) is present in H, so $f(u) = g(u_0) \neq g(v_0) = f(v)$, and f is a 3-coloring of G. □

Lemma 9. *A graph $H = (V, E)$ with $|V| = 3n$ vertices and $|E| \leq 3n^2$ edges for $n > 1$ satisfies $\lambda_2(H) \geq 2n$ if and only if H is (isomorphic to) $K_{n,n,n}$.*

Proof. First, we compute the algebraic connectivity of $K_{n,n,n}$. The complementary graph of $K_{n,n,n}$ consists of three connected components, each of which is a complete graph on n vertices. By Properties 4 and 5, the largest eigenvalue of the Laplacian of the complementary graph is n. Property 3 now implies that $\lambda_2(K_{n,n,n}) = 3n - n = 2n$.

Now, let H be a graph with $3n$ vertices and at most $3n^2$ edges for $n > 1$ that has algebraic connectivity at least $2n$. Since H is not a complete graph, Property 6 implies that every vertex in H must have degree at least $2n$. On the other hand, the average degree of vertices in H is at most $2(3n^2)/(3n) = 2n$. So, H must be $(2n)$-regular.

Consider the complementary graph $H^c = (V, E^c)$ of H, which is $(n-1)$-regular. By Property 3, $\lambda_{\max}(H^c) \leq 3n - 2n = n$. If H^c had a connected component with more than n vertices, then Property 7 would imply that $\lambda_{\max}(H^c) > n$. We conclude from this contradiction that every connected component in H^c has at most n vertices.

The fact that H^c is $(n-1)$-regular now implies that every connected component of H^c must be a complete graph on n vertices. Thus, H^c consists of three connected components, each of which is a complete graph on n vertices, and the original graph H is isomorphic to $K_{n,n,n}$. □

This NP-hardness result suggests several directions for future research on the maximum algebraic connectivity augmentation problem. The natural next open questions are those of whether hardness of approximation results, such as APX-hardness, can be established, and whether approximation algorithms with worst-case guarantees can be found.

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