Make sure to provide justification for your answers. This includes labeling all of your plots (title, axes, legend, etc.) and explaining what is shown in the plots. Otherwise, you will lose points.

In this homework assignment, we will review kinematics and dynamics concepts from CS223A and then set up the simulation environment (SAI) that will be used for subsequent homework assignments and the final project.

1. Consider the RPR manipulator below. Assume this robot is massless except for the end effector, which is a point mass $m_4 = 1.0$ kg. Note that the joint angle $\theta_3$ is negative in the configuration drawn below.

![Diagram of RPR manipulator]

(a) Find the position $^0P_4$ of the end effector expressed in frame $\{0\}$. *Hint:* Avoid DH parameters and use geometric intuition instead. Break up the projections by finding $^2P_4$ before $^0P_4$.

**Solution:**

We will use the shorthand notation $c_i := \cos(\theta_i)$, $s_i := \sin(\theta_i)$.

\[
^0P_4^h = ^0T_2^2P_4^h = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -L_4s_3 \\ 0 \\ L_4c_3 \\ 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} -L_4c_3s_3 \\ -L_4s_1s_3 \\ L_4c_3 + d_2 \\ 1 \end{bmatrix}
\]

\[
\Rightarrow ^0P_4 = \begin{bmatrix} -L_4c_3s_3 \\ -L_4s_1s_3 \\ L_4c_3 + d_2 \end{bmatrix}
\]
(b) Find the orientation $^0_4R$ of the end effector frame $\{4\}$ expressed in frame $\{0\}$. *Hint:* Break up the projections by finding $^2_4R$ before $^0_4R$.

**Solution:**

\[
^0_4R = ^0_2R^2_4R \\
= \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 \\ 0 & 0 & -1 \\ s_3 & c_3 & 0 \end{bmatrix} \\
= \begin{bmatrix} c_1c_3 & -c_1s_3 & s_1 \\ s_1c_3 & -s_1s_3 & -c_1 \\ s_3 & c_3 & 0 \end{bmatrix}
\]

(c) Find the linear Jacobian $^0J_v$ of the end effector expressed in frame $\{0\}$.

**Solution:**

\[
^0J_v = \begin{bmatrix} \frac{\partial^0P}{\partial\theta_1} & \frac{\partial^0P}{\partial\theta_2} & \frac{\partial^0P}{\partial\theta_3} \end{bmatrix} \\
= \begin{bmatrix} L_4s_1s_3 & 0 & -L_4c_1c_3 \\ -L_4c_1s_3 & 0 & -L_4s_1c_3 \\ 0 & 1 & -L_4s_3 \end{bmatrix}
\]

(d) Find the angular Jacobian $^0J_\omega$ of the end effector expressed in frame $\{0\}$.

**Solution:**

Here, we use the notation

\[
\bar{\epsilon}_i = \begin{cases} 
1 & \text{\textit{i}th joint is revolute} \\
0 & \text{\textit{i}th joint is prismatic}
\end{cases}
\]

and $Z_i$ represents the axis of rotation for revolute joints and axis of translation for prismatic joints.

\[
^0J_\omega = [\bar{\epsilon}_1 \ 0Z_1 \ \bar{\epsilon}_2 \ 0Z_2 \ \bar{\epsilon}_3 \ 0Z_3] \\
= \begin{bmatrix} 0 & 0 & s_1 \\ 0 & 0 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}
\]

Note that the third column in $^0J_\omega$ corresponds to the third column in $^0R$ from Part (b).
(e) Find the linear singularities of this robot. For each singularity you find, draw the robot in the singular configuration and specify the singular direction. Avoid taking the determinant and use intuition to identify the singularities.

Solution:

There are linear singularities when $\theta_3 = k\pi$ and $\theta_3 = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$.

i. An example of a configuration in which $\theta_3 = k\pi$ is shown below. In this case, the singular direction is $Z_4$.

ii. An example of a configuration in which $\theta_3 = \frac{\pi}{2} + k\pi$ is shown below. In this case, the singular direction is $Y_4$. 
(f) Write out $^0J_v$ for the singular configurations you drew above. For each configuration, explain why these are singularities in terms of joint motion. For each singular configuration, what is the singular direction expressed in frame \{0\} when $\theta_1 = 0^\circ$? When $\theta_1 = 90^\circ$?

**Solution:**

i. When $\theta_3 = 0^\circ$, we get the following Jacobian:

$$^0J_v = \begin{bmatrix} 0 & 0 & -L_4c_1 \\ 0 & 0 & -L_4s_1 \\ 0 & 1 & 0 \end{bmatrix}$$

The first column is 0, indicating that the first joint loses the ability to produce velocities at the end effector. When $\theta_1 = 0^\circ$, the Jacobian becomes:

$$^0J_v = \begin{bmatrix} 0 & 0 & -L_4 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The second row is 0, indicating that the singular direction is along $Y_0$. Similarly, when $\theta_1 = 90^\circ$, the Jacobian becomes:

$$^0J_v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_4 \\ 0 & 1 & 0 \end{bmatrix}$$

The first row is 0, indicating that the singular direction is along $X_0$.

ii. When $\theta_3 = 90^\circ$, we get the following Jacobian:

$$^0J_v = \begin{bmatrix} L_4s_1 & 0 & 0 \\ -L_4c_1 & 0 & 0 \\ 0 & 1 & -L_4 \end{bmatrix}$$

The second and third columns are multiples of each other, indicating that the second and third joints produce velocities at the end effector that are along the same direction. When $\theta_1 = 0^\circ$, the Jacobian becomes:

$$^0J_v = \begin{bmatrix} 0 & 0 & 0 \\ -L_4 & 0 & 0 \\ 0 & 1 & -L_4 \end{bmatrix}$$

The first row is 0, indicating that the singular direction is along $X_0$. Similarly, when $\theta_1 = 90^\circ$, the Jacobian becomes:

$$^0J_v = \begin{bmatrix} L_4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -L_4 \end{bmatrix}$$

The second row is 0, indicating that the singular direction is along $Y_0$.  


(g) Find the joint space mass matrix $M$ for this manipulator. Recall that all the robot is massless (i.e., no mass or inertia on any links or joints) except for the end effector, which is a point mass $m_4 = 1.0$ kg.

Solution:
Since all the links before the end effector are massless, we only need to consider the last mass, $m_4$. Because it is a point mass, and we consider at the center of this point mass, it has zero inertia. Therefore, we can construct the mass matrix with:

$$
M = \sum_{i=1}^{4} (m_i J_v^T v_i + J_{\omega i}^T \omega_i I_c) = m_4 J_{v_4}^T v_4
$$

$$
= m_4 \begin{bmatrix}
L_4 s_1 s_3 & -L_4 c_1 c_3 \\
-L_4 c_1 s_3 & -L_4 s_1 c_3 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
L_4 s_1 s_3 & 0 & -L_4 c_1 c_3 \\
0 & 1 & -L_4 s_3 \\
-2 L_1^2 s_1 s_3 c_3 & -2 L_1^2 s_1 c_1 c_3 c_3 & -L_4 s_3 \\
0 & 1
\end{bmatrix}
$$

$$
= m_4 \begin{bmatrix}
L_4^2 s_3 (s_1^2 + c_3^2) & 0 & 0 \\
0 & 1 & -L_4 s_3 \\
0 & -L_4 s_3 & L_4^2 (c_3^2 + s_3^2)
\end{bmatrix}
$$

$$
= m_4 \begin{bmatrix}
L_4^2 s_3^2 & 0 & 0 \\
0 & 1 & -L_4 s_3 \\
0 & -L_4 s_3 & L_4^2 (c_3^2 + s_3^2)
\end{bmatrix}
$$

(h) Find the gravity vector $G$. Assume acceleration due to gravity is $[0, 0, -g]^T$, where $g = 9.81$ m s$^{-2}$.

Solution:

$$
G = -\sum_{i=1}^{4} J_{v_i}^T (m_i g)
$$

$$
= -J_{v_4}^T (m_4 g)
$$

$$
= -m_4 \begin{bmatrix}
L_4 s_1 s_3 & 0 & -L_4 c_1 c_3 \\
-2 L_1^2 s_1 s_3 c_3 & -2 L_1^2 s_1 c_1 c_3 c_3 & -L_4 s_3 \\
0 & 1 & -L_4 s_3
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
-g
\end{bmatrix}
$$

$$
= m_4 g \begin{bmatrix}
0 \\
1 \\
-L_4 s_3
\end{bmatrix}
$$
2. Now, we will run this robot in simulation. We will let the length \( L_4 = 1.0 \) \( \text{m} \). Follow the setup instructions from the attached PDF and the repository [https://github.com/manips-sai-org/cs225a](https://github.com/manips-sai-org/cs225a) to get SAI installed on your system.

(a) As we mentioned in class, the urdf specification of robots often does not follow the standard DH conventions. This is the case for the prbot.urdf provided in the folder hw0. Complete the following schematic by placing the frames that correspond to the model described by the urdf file. *Hint:* Keep in mind that, in the urdf file, the kinematic structure is described by the joints only.

Solution:

(b) What is the position of the end effector in frame \( \{3\}, \; ^3P_4 \)? Replace the value of “ee pos in link” in the hw0.cpp file with this value.

Solution:

The position of the end-effector expressed in frame \( \{3\} \) is the distance from the origin of frame \( \{3\} \) to the end-effector expressed in frame \( \{3\} \)'s coordinate basis vectors. Since the end-effector is \( L_4 \) away from the origin of frame \( \{3\} \) along \( Z_3 \), we get the following position vector:

\[
^3P_4 = \begin{bmatrix} 0 \\ 0 \\ L_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.0 \text{ m} \end{bmatrix}
\]

(c) Find the end effector position in frame \( \{0\}, \; ^0P_4 \), for the two configurations below. Are the results consistent with the expression you found in 1 (a)? What is different? Explain.

i. \( \theta_1 = 0^\circ, \; d_2 = 1.0 \text{ m}, \; \theta_3 = -90^\circ \).

ii. \( \theta_1 = 90^\circ, \; d_2 = 1.0 \text{ m}, \; \theta_3 = -90^\circ \)

Solution:

i. In this configuration, we get \( ^0P_4 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T \). This is not consistent with the expression we found in 1 (a), which would give us \( ^0P_4 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \). In this problem, the \( X_1 \) axis is the \(-Y_1\) axis from 1 (a). Similarly, in this problem, the \( Y_1 \) axis is the \( X_1 \) axis from 1 (a).
ii. In this configuration, we get $0\mathbf{P}_4 = [-1 \ 0 \ 1]^T$. This is not consistent with the expression we found in 1 (a), which would give us $0\mathbf{P}_4 = [0 \ 1 \ 0]^T$. In this problem, the $X_1$ axis is the $-Y_1$ axis from 1 (a). Similarly, in this problem, the $Y_1$ axis is the $X_1$ axis from 1 (a).

(d) Find the simulated linear Jacobian $0\mathbf{J}_v$ for the two configurations considered in 2 (c). Are the results consistent with the expression you found in 1 (c)? What is different? Explain.

**Solution:**

i. In this configuration, we get $0\mathbf{J}_v = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. This is not consistent with the expression we found in 1 (a), which would give us $0\mathbf{J}_v = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. In this problem, the $X_1$ axis is the $-Y_1$ axis from 1 (a). Similarly, in this problem, the $Y_1$ axis is the $X_1$ axis from 1 (a).

ii. In this configuration, we get $0\mathbf{J}_v = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. This is not consistent with the expression we found in 1 (a), which would give us $0\mathbf{J}_v = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. In this problem, the $X_1$ axis is the $-Y_1$ axis from 1 (a). Similarly, in this problem, the $Y_1$ axis is the $X_1$ axis from 1 (a).

(e) For the following plots, output the desired mass matrix values to a file and use a program like MATLAB or Excel for plotting. For each, compare your results to 1 (g) and explain physically why each configuration sweep produces your results.

i. Set $\theta_1 = 0^\circ$ and $d_2 = 1.0 \text{ m}$. Plot $m_{11}$, $m_{22}$, and $m_{33}$ as $\theta_3$ varies from $-90^\circ$ to $90^\circ$.

ii. Set $\theta_1 = 0^\circ$ and $\theta_3 = 0^\circ$. Plot $m_{11}$, $m_{22}$, and $m_{33}$ as $d_2$ varies from $0.0 \text{ m}$ to $2.0 \text{ m}$.

**Solution:**

Recalling our expression for the joint space mass matrix from 1 (g), we have the following:

$$M = m_4 \begin{bmatrix} L_4^2 s_3^2 & 0 & 0 \\ 0 & 1 & -L_4 s_3 \\ 0 & -L_4 s_3 & L_4^2 \end{bmatrix}.$$

When we derived the joint space mass matrix, we assumed that all links are massless. However, looking at the `rprbot.urdf` file, we see that each link (except for the last one) has a mass of 0.01 kg. Factoring this into our expression for the joint space mass matrix, we get

$$M = \begin{bmatrix} m_4 L_4^2 s_3^2 & 0 & 0 \\ 0 & m_4 + 0.01 & -m_4 L_4 s_3 \\ 0 & -m_4 L_4 s_3 & m_4 L_4^2 \end{bmatrix}.$$

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1If you’re feeling ambitious, you can output the keys to Redis and create a Python script to catch the keys and plot them with Matplotlib. We will be using Redis heavily later in the course, so it’ll be good to familiarize yourself with it.
In our problem, we specify that $m_4 = 1.0\text{ kg}$ and $L_4 = 1.0\text{ m}$. Therefore, we get the following values for the diagonal joint space mass matrix elements (ignoring units):

\[
\begin{align*}
m_{11} & = s_3^2; \\
m_{22} & = 1.01; \\
m_{33} & = 1.
\end{align*}
\]

Therefore, when we vary $\theta_3$, the only diagonal element that will change is $m_{11}$. Physically, due to the robot’s design, the second joint always feels the same effective mass (i.e., the cumulative mass of the links that come after it). The third joint’s effective mass is also independent of the robot’s configuration.

Similarly, when we vary $d_2$, all of the diagonal elements in the joint space mass matrix remain constant. The value of $m_{33}$ is fixed at 0 because $\sin^2(0) = 0$. The plots demonstrating these effects are shown below.
(f) Now, produce the same plots for $G$. For each, compare your results to 1 (h) and explain physically why each configuration sweep produces your results.

i. Set $\theta_1 = 0^\circ$ and $d_2 = 1.0$ m. Plot $G$ as $\theta_3$ varies from $-90^\circ$ to $90^\circ$.

ii. Set $\theta_1 = 0^\circ$ and $\theta_3 = 0^\circ$. Plot $G$ as $d_2$ varies from 0.0 m to 2.0 m.

**Solution:**

Recalling our expression for the joint space gravity vector from 1 (h), we have the following:

$$G = m_4 g \begin{bmatrix} 0 \\ 1 \\ -L_4 s_3 \end{bmatrix}.$$  

When we derived the joint space gravity vector, we assumed that all links are massless. However, looking at the `rprbot.urdf` file, we see that each link (except for the last one) has a mass of 0.01 kg. Factoring this into our expression for the joint space gravity vector, we get

$$G = g \begin{bmatrix} 0 \\ m_4 + 0.01 \\ -m_4 L_4 s_3 \end{bmatrix}.$$  

In our problem, we specify that $m_4 = 1.0$ kg, $L_4 = 1.0$ m, and $g = 9.81$ m s$^{-2}$. Therefore, we get the following values for the joint space gravity vector elements (ignoring units):

$$G_1 = 0;$$
$$G_2 = 9.81 \cdot 1.01 = 9.9081;$$
$$G_3 = -9.81 \cdot s_3.$$  

Therefore, when we vary $\theta_3$, the only element that will change is $G_3$. Physically, the first joint is revolute and aligned with the direction of gravity, so it never feels any gravitational torque. Similarly, the second joint is prismatic and also aligned with the direction of gravity, so it always feels the same gravitational torque (i.e., the cumulative mass of the links that come after it multiplied by the gravitational acceleration). In contrast, the gravitational torque felt by the third joint depends on the third joint’s configuration. When the third joint aligns the last link with the direction of gravity (i.e., when $\theta_3 = 0, \pi$), then the third joint feels no gravitational torque. However, when $\theta_3 = \frac{\pi}{2}, \frac{3\pi}{2}$, the gravitational torque felt by the third joint is maximized and equal to the torque produced by the point mass $m_4$ multiplied by the moment arm of $L_4$.

With the same reasoning as before, when we vary $d_2$, the gravitational torques felt by the first two joints are constant (and equal to the values described before). Furthermore, when we vary $d_2$, the gravitational torque felt by the third joint is constant and fixed by the value of $\theta_3 = 0$, which also makes the gravitational torque felt by the third joint equal to 0 for the same reasoning as before. The plots demonstrating these effects are shown on the next page.
(g) **Extra credit:** Modify the rprbot.urdf file to add a prismatic joint between frame \{3\} and frame \{4\}. Plot the gravity vector for this new robot when $\theta_1 = 0^\circ$, $d_2 = 1.0\text{ m}$, $\theta_3 = 45^\circ$ and $d_4$ varies from 0 m to 1 m.

**Solution:**

With the new robot, we have the following position of the end-effector expressed in frame \{0\}:

$$
0P_4 = \begin{bmatrix}
-(l_4 + d_4)c_1s_3 \\
-(l_4 + d_4)s_1s_3 \\
d_2 - (l_4 + d_4)c_3
\end{bmatrix}.
$$

Now, the linear Jacobian of the end-effector expressed in \{0\} is

$$
0J_v = \begin{bmatrix}
(L_4 + d_4)s_1s_3 & 0 & -(L_4 + d_4)c_1c_3 & -c_1s_3 \\
-(L_4 + d_4)c_1s_3 & 0 & -(L_4 + d_4)s_1c_3 & -s_1s_3 \\
0 & 1 & -(L_4 + d_4)s_3 & c_3
\end{bmatrix}.
$$

For the specified robot configuration, the linear Jacobian of the end-effector expressed in \{0\} becomes

$$
0J_v = \begin{bmatrix}
0 & 0 & -(1 + d_4)\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
-(1 + d_4)\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\
0 & 1 & -(1 + d_4)\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{bmatrix}.
$$

Now, the gravitational torques become

$$
G = -0J_v^T m_4 \vec{g}
$$

$$
= m_4 \begin{bmatrix}
0 & -(1 + d_4)\frac{\sqrt{2}}{2} & 0 \\
0 & 0 & 1 \\
-(1 + d_4)\frac{\sqrt{2}}{2} & 0 & -(1 + d_4)\frac{\sqrt{2}}{2}
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
-9.81\text{ m s}^{-2}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 \\
9.81 \\
-6.94(1 + d_4)
\end{bmatrix}.
$$

Accounting for the links that have a mass of 0.01 kg in the urdf file (as opposed to massless links), we get the corrected vector of gravitational torques:

$$
G = \begin{bmatrix}
0 \\
9.81 \cdot 1.02 \\
-6.94(1 + d_4)
\end{bmatrix},
$$

$$
= \begin{bmatrix}
0 \\
10.0062 \\
-6.94(1 + d_4)
\end{bmatrix}.
$$

Therefore, $G_3$ is the only value that changes with $d_4$. This behavior, along with the constant values for the other gravitational torques, is shown on the next page. The modified rprbot.urdf and hw0.cpp files are provided in the hw0_solution directory in the cs225a git repository.
3. Submit your SAI code (required: hw0.cpp file; extra credit: modified rprbot.urdf).