Experimental Robotics

CS225A

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Menu

- Kinematics Dynamics
  - Jacobians Inverses
  - Task Representations
  - Equations of Motion
- Operational Space Control
  - Dynamic Models
  - Control Modalities
  - Compliance Force Control
- Redundant Robots
  - Posture Null Space
  - Dynamic Behavior
  - Whole-Body Control

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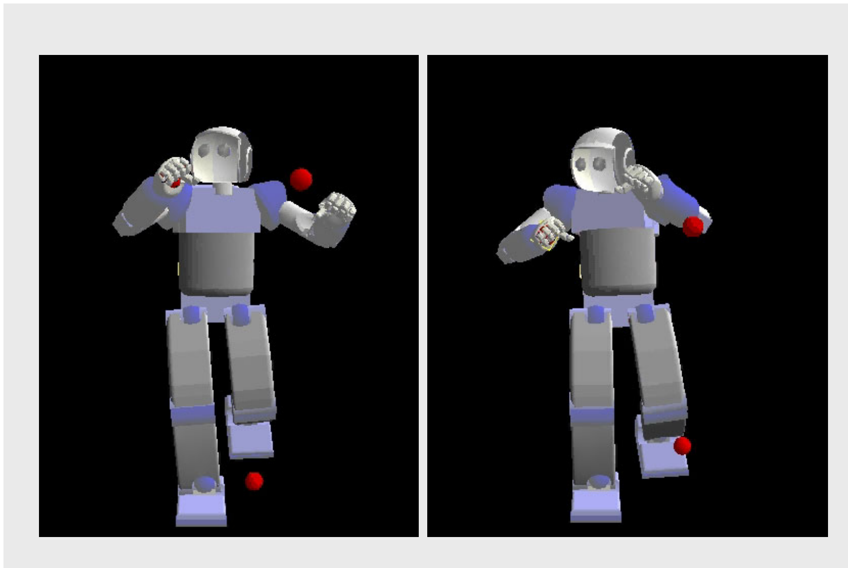

ASIMO

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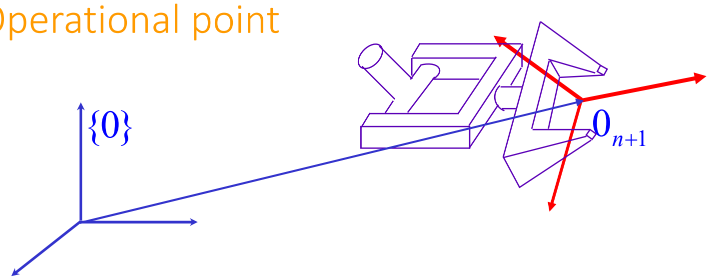
Operational Coordinates

O_{n+1} : Operational point



A set $x_1, x_2, \ldots, x_{m_0}$
of m_0 independent configuration parameters

m_0 : number of degrees of freedom of the end-effector.

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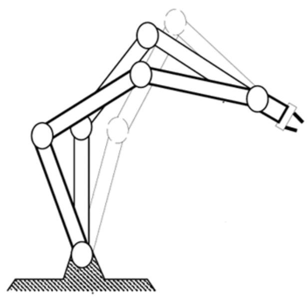
Redundancy



A robot is said to be redundant if

$$n > m_0$$

Degrees of redundancy: $n - m_0$

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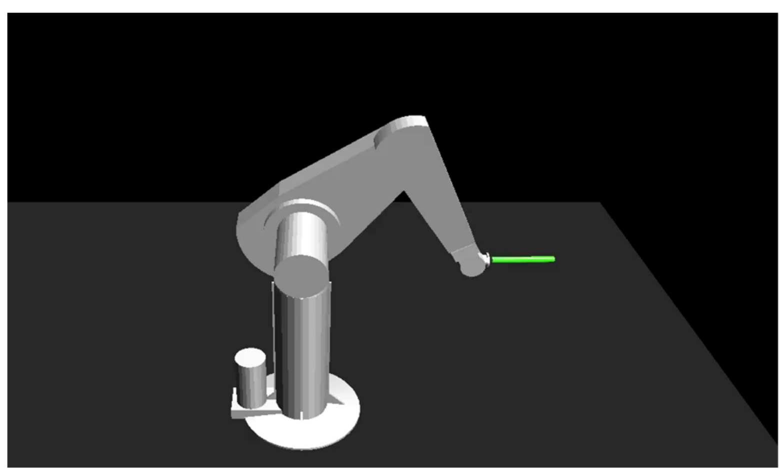
Redundancy



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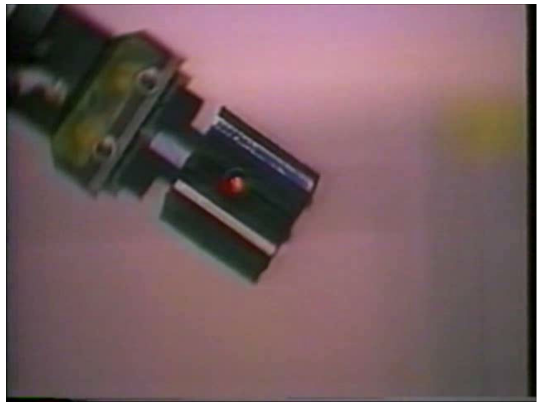
Task Redundancy



$$n > m_{task}$$

$n - m_{task}$: degrees of redundancy/task

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- Forward Kinematics
- Inverse Kinematics

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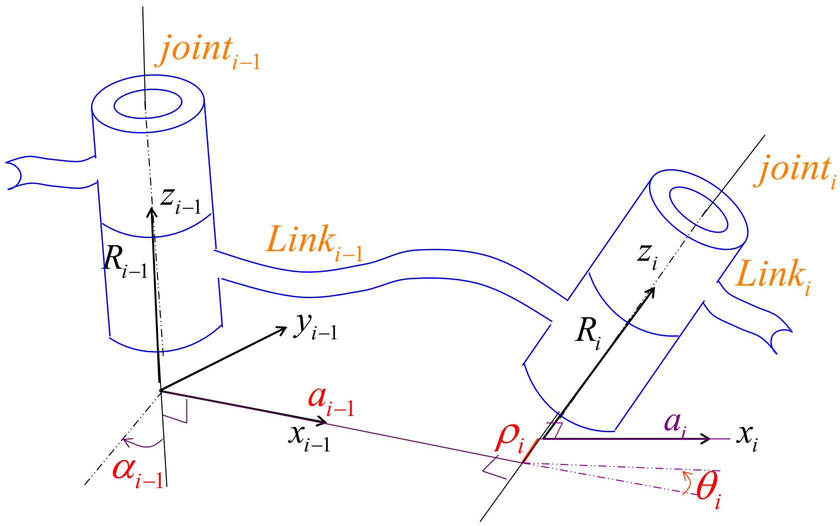
Denavit-Hartenberg (DH) Parameters



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Homogeneous Transformation

$$p_{(i-1)} = T_{(i-1)i}\ p_i$$

$$T_{(i-1)i} = T_{(i-1)i}\left(\alpha_{i-1}, a_{i-1}, \theta_i, \rho_i\right)$$

$$T_{(i-1)i} = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{(i-1)} \\ \sin\theta_i \cos\alpha_{(i-1)} & \cos\theta_i \cos\alpha_{(i-1)} & -\sin\alpha_{(i-1)} & -\rho_i \sin\alpha_{(i-1)} \\ \sin\theta_i \sin\alpha_{(i-1)} & \cos\theta_i \sin\alpha_{(i-1)} & \cos\alpha_{(i-1)} & \rho_i \cos\alpha_{(i-1)} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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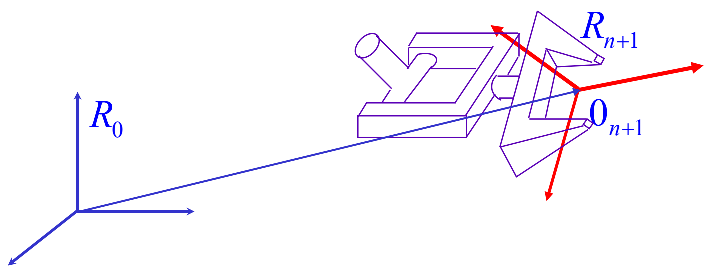
Geometric Model
Forward Kinematics

$$T_{0(n+1)}(q) = T_{01}(q_1)T_{12}(q_2)\ldots T_{(n-1)n}(q_n)T_{n(n+1)}$$



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Geometric Model
Forward Kinematics

$$T_{0(n+1)}(q) = \begin{pmatrix} R_{o(n+1)}(q) & \rho_{o(n+1)}(q) \\ 0 & 1 \end{pmatrix}$$

m equations

$$x = G(q) \qquad x = \begin{pmatrix} x_p(q) \\ x_r(q) \end{pmatrix}$$

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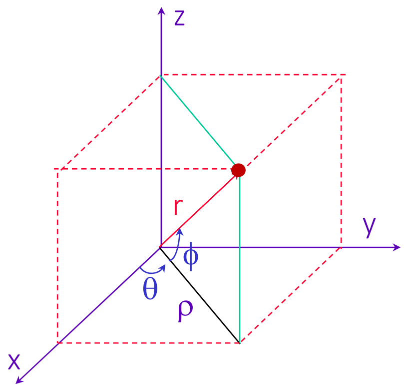
Position Representations



Cartesian: (x, y, z)

Cylindrical: (ρ, θ, z)

Spherical: (r, θ, ϕ)

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Rotation Representations

Rotation Matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix}$$

Direction Cosines

$$x_r = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}_{(9x1)}$$

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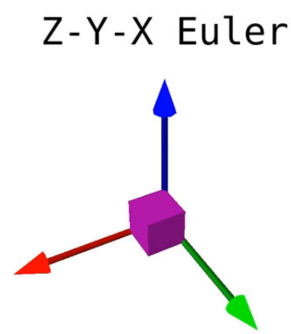
Three-Angles



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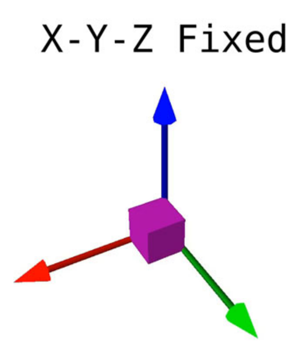
Three-Angles



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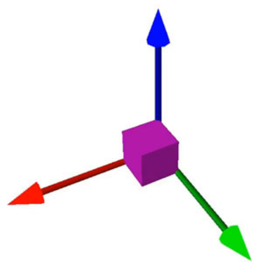
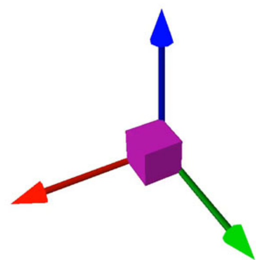
Three-Angles

Z-Y-X Euler



X-Y-Z Fixed



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Euler Parameters

Quaternion

Rotations are
Product of two plane symmetries

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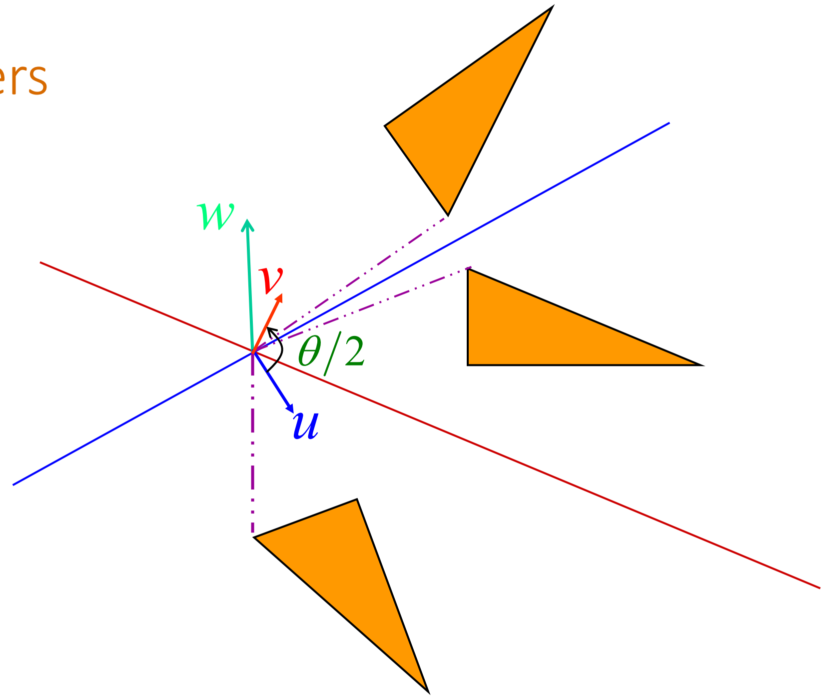
Euler Parameters

$$u \cdot v = \cos \theta/2$$

$$u \times v = w \sin \theta/2$$



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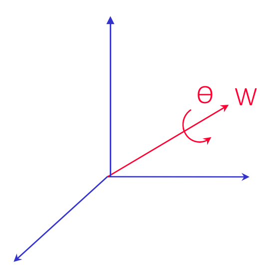
Euler Parameters

$$\lambda_0 = \cos \theta/2 \; ;$$
$$\lambda_1 = w_1 \sin \theta/2 \; ;$$
$$\lambda_2 = w_2 \sin \theta/2 \; ;$$
$$\lambda_3 = w_3 \sin \theta/2 \; ;$$



Normality condition

$$\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$$

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Rotation Matrix

$$R(q)=\begin{pmatrix} 2(\lambda_0^2+\lambda_1^2)-1 & 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & 2(\lambda_1\lambda_3+\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2+\lambda_0\lambda_3) & 2(\lambda_0^2+\lambda_2^2)-1 & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3-\lambda_0\lambda_2) & 2(\lambda_2\lambda_3+\lambda_0\lambda_1) & 2(\lambda_0^2+\lambda_3^2)-1 \end{pmatrix}$$

$$R=\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

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Euler Parameters

Lemma: For all rotations, at least one of the Euler Parameters has a magnitude larger than or equal to 1/2.

Algorithm

	$\|\lambda_0((t_{(i-1)})\|$
$\lambda_0(t_i)$	$\Delta_0/4$
$\lambda_1(t_i)$	$(s_{32}-s_{23})/\Delta_0$
$\lambda_2(t_i)$	$(s_{13}-s_{31})/\Delta_0$
$\lambda_3(t_i)$	$(s_{21}-s_{12})/\Delta_0$

with

$$\Delta_0 = 2\text{sgn}\left(\lambda_0(t_{(i-1)})\right)\sqrt{s_{11}+s_{22}+s_{33}+1};$$

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Jacobian for X

Given a representation $x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$

$$\dot{x} = J_x(q)\ \dot{q}$$

$$J_x(q) = E(x)\ J_0(q)$$

Basic Jacobian $\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q)\ \dot{q}$

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Jacobian and Basic Jacobian

$$J = \begin{pmatrix} J_{XP} \\ \hline J_{XR} \end{pmatrix} = \begin{pmatrix} E_P & 0 \\ \hline 0 & E_R \end{pmatrix} \begin{pmatrix} J_v \\ \hline J_w \end{pmatrix}$$

$$J(q) = E(X)\ J_0(q)$$

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q)\ \dot{q}$$

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Position Representations: $E_P(x)$

Cartesian Coordinates (x, y, z)

$$E_P(x_p) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \quad y \quad z)^T = (\rho\cos\theta \quad \rho\sin\theta \quad z)^T$

$$E_P(x_p) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Position Representations: $E_P(x)$

Spherical Coordinates (ρ, θ, ϕ)

Using

$$(x \quad y \quad z)^T = (\rho\cos\theta\sin\phi \quad \rho\sin\theta\sin\phi \quad \rho\cos\theta)^T$$

$$E_P(x_p) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & \_\sin\phi/\rho \end{pmatrix}$$

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Rotation Representations: $E_R(x)$

Euler Angles

$$x_R = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}; E_R(x_R) = \begin{pmatrix} -\frac{s\alpha . c\beta}{s\beta} & \frac{c\alpha . c\beta}{s\beta} & 1 \\ c\alpha & s\alpha & 0 \\ \frac{s\alpha}{s\beta} & -\frac{c\alpha}{s\beta} & 0 \end{pmatrix}$$

Singularity of the representation
for $\beta = k\pi$

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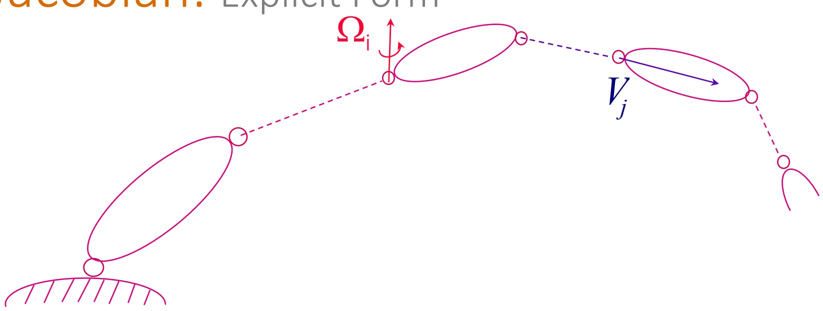
The Jacobian: Explicit Form



Revolute Joint $\Omega_i = Z_i \dot{q}_i$

Prismatic Joint $V_i = Z_i \dot{q}_i$

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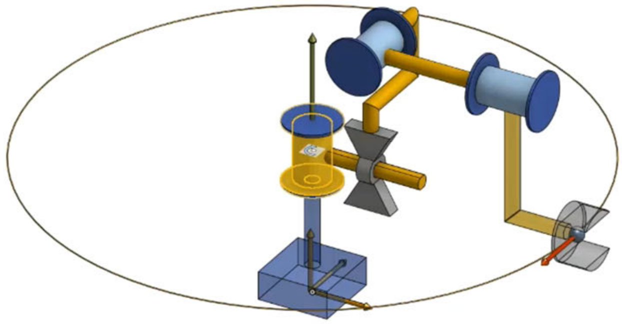
Example



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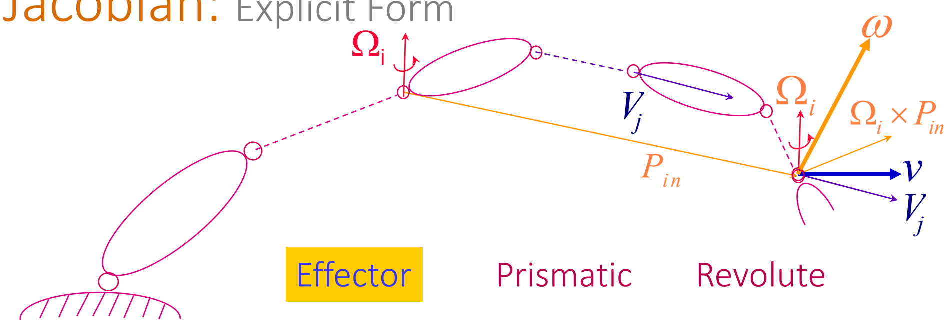
The Jacobian: Explicit Form



Effector	Prismatic	Revolute
Linear Vel:	V_j	$\Omega_i \times P_{in}$
Angular Vel:	none	Ω_i

Effector Linear Velocity $$v = \sum_{i=1}^{n} [\epsilon_i V_i + \bar{\epsilon}_i (\Omega_i \times P_{in})] \Leftarrow V_i = Z_i \dot{q}_i$$

Effector Angular Velocity $$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i \Omega_i \Leftarrow \Omega_i = Z_i \dot{q}_i$$

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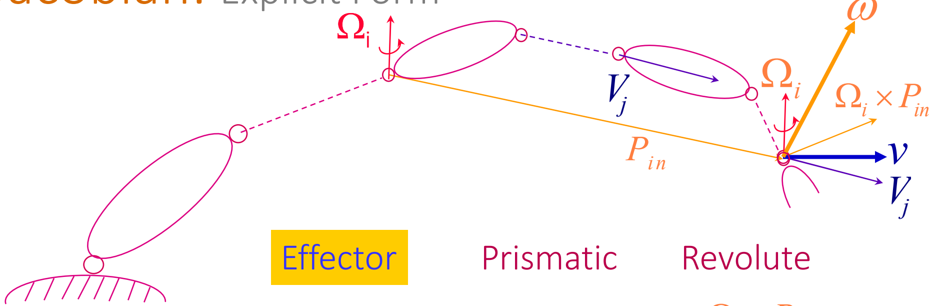
The Jacobian: Explicit Form



Effector	Prismatic	Revolute
Linear Vel:	V_j	$\Omega_i \times P_{in}$
Angular Vel:	none	Ω_i

Effector Linear Velocity $v = \sum_{i=1}^{n} [\epsilon_i Z_i + \bar{\epsilon}_i (Z_i \times P_{in})] \dot{q}_i \Leftarrow V_i = Z_i \dot{q}_i$

Effector Angular Velocity $\omega = \sum_{i=1}^{n} (\bar{\epsilon}_i Z_i) \dot{q}_i \Leftarrow \Omega_i = Z_i \dot{q}_i$

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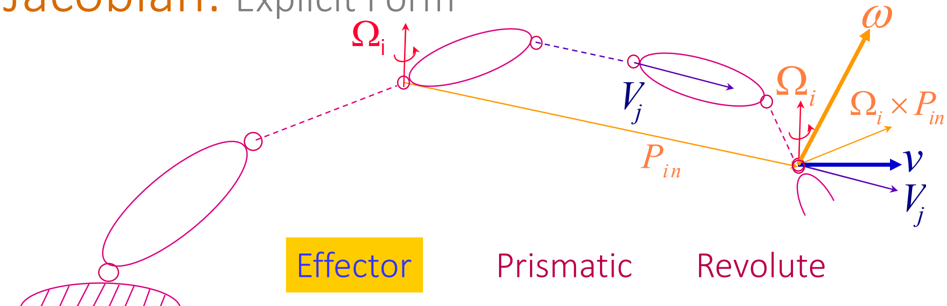
The Jacobian: Explicit Form



Effector	Prismatic	Revolute
Linear Vel:	V_j	$\Omega_i \times P_{in}$
Angular Vel:	none	Ω_i

Effector Linear Velocity $v = \sum_{i=1}^{n} (\epsilon_i Z_i + \bar{\epsilon}_i \hat{Z}_i P_{in}) \dot{q}_i \Leftarrow V_i = Z_i \dot{q}_i$

Effector Angular Velocity $\omega = \sum_{i=1}^{n} (\bar{\epsilon}_i Z_i) \dot{q}_i \Leftarrow \Omega_i = Z_i \dot{q}_i$

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The Jacobian: Explicit Form

$$v=\begin{bmatrix} \epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) & \epsilon_2 Z_2 + \bar{\epsilon}_2 (Z_2 \times P_{2n}) & \cdots \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

$$\omega = \begin{bmatrix} \bar{\epsilon}_1 Z_1 & \bar{\epsilon}_2 Z_2 & \cdots & \bar{\epsilon}_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{\omega = J_\omega \dot{q}}$$

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The Jacobian: Explicit Form

$$v=\begin{bmatrix} \epsilon_1 Z_1 + \bar{\epsilon}_1 \hat{Z}_1 P_{1n} & \epsilon_2 Z_2 + \bar{\epsilon}_2 \hat{Z}_2 P_{2n} & \cdots \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

$$\omega = \begin{bmatrix} \bar{\epsilon}_1 Z_1 & \bar{\epsilon}_2 Z_2 & \cdots & \bar{\epsilon}_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{\omega = J_\omega \dot{q}}$$

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The Jacobian: Explicit Form

$$J = \begin{pmatrix} J_v \\ \hline J_w \end{pmatrix}$$

Matrix J_v (direct differentiation)

$$v = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{x}_P = \frac{\partial x_P}{\partial q_1}.\dot{q}_1 + \frac{\partial x_P}{\partial q_2}.\dot{q}_2 + \cdots + \frac{\partial x_P}{\partial q_n}.\dot{q}_n$$

$$J_v = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \end{pmatrix}$$

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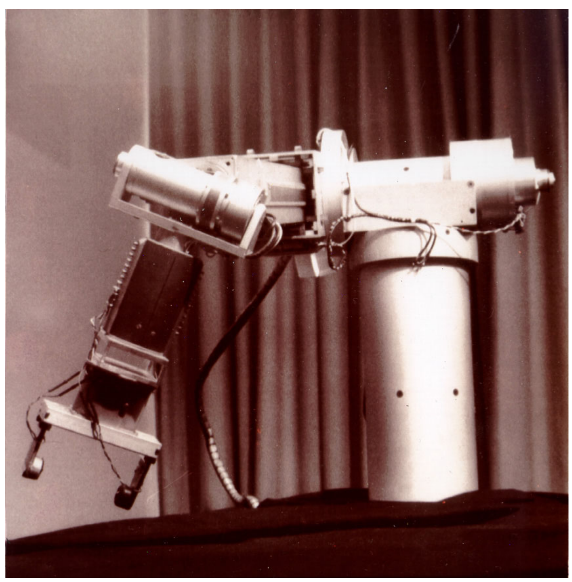
Stanford Scheinman Arm



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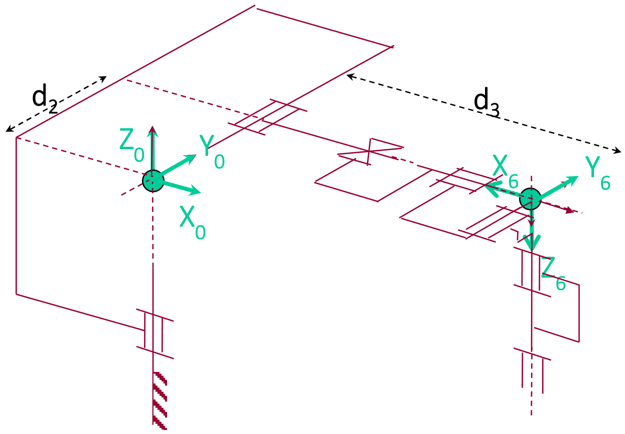

$$J = \left(\begin{array}{c:c:c:c:c:c} Z_1 \times P_{13} & Z_2 \times P_{23} & Z_3 & 0 & 0 & 0 \\ \hdashline Z_1 & Z_2 & 0 & Z_4 & Z_5 & Z_6 \end{array} \right)$$

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Stanford Scheinman Arm Jacobian

$$J = \begin{bmatrix} -c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\ -s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\ 0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\ 0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ 0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ 1 & 0 & 0 & c_2 & s_2 s_4 & -s_2 c_4 s_5 + c_5 c_2 \end{bmatrix}$$

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Velocity/Force Duality

$$\dot{x} = J \, \dot{\theta}$$

$$\tau = J^T F$$

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Instantaneous
Inverse Kinematics

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Linearized Kinematic Model

$$\delta x = J(q)\,\delta q$$

Resolved Motion-Rate (Whitney 1972)

$$\delta q = J^{-1}(q)\,\delta x$$

$$q^{+} = q + J^{-1}(q)\,\delta x$$

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Inverse of the Jacobian

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = J_{2\times 2} \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix}$$

Inverse Jacobian

$$\begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix} = J^{-1}_{2\times 2} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

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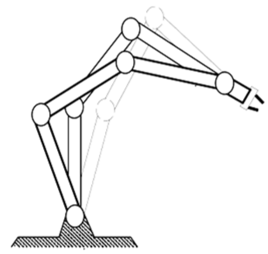
Redundancy

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = J_{2\times3} \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \\ \Delta q_3 \end{pmatrix}$$



Generalized Inverse

$$\begin{pmatrix} \Delta q_1 \\ \Delta q_2 \\ \Delta q_3 \end{pmatrix} = J^{\#}_{3\times2} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} + \left[I - J^{\#} J \right]_{3\times3} \begin{bmatrix} \Delta q_1 \\ \Delta q_2 \\ \Delta q_3 \end{bmatrix}$$

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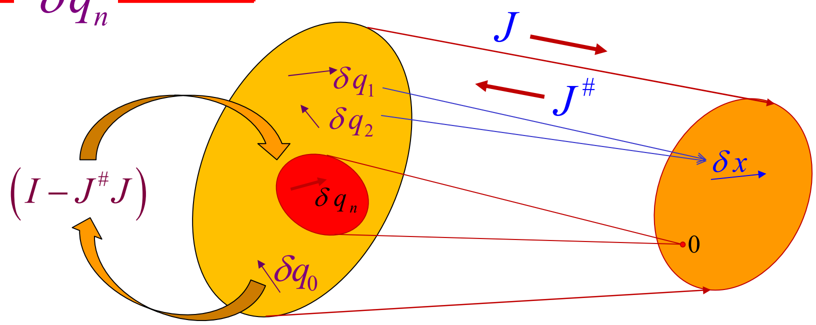
General Solution

$$\delta q = J^{\#}\,\delta x + \underbrace{\left[I - J^{\#} J \right] \delta q_0}_{\delta q_n}$$



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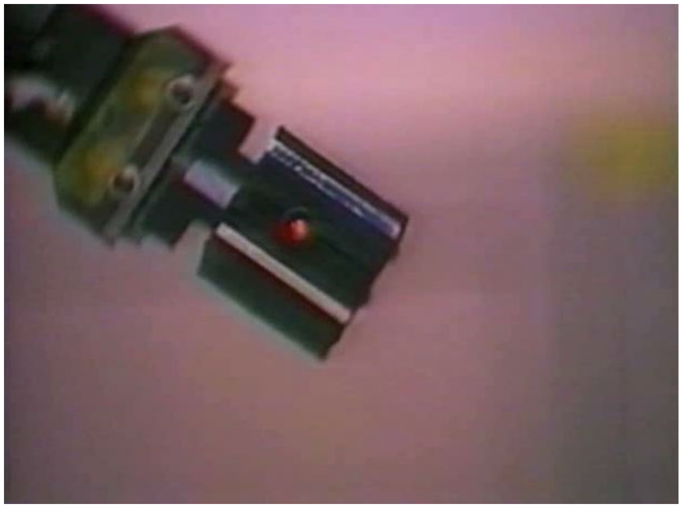

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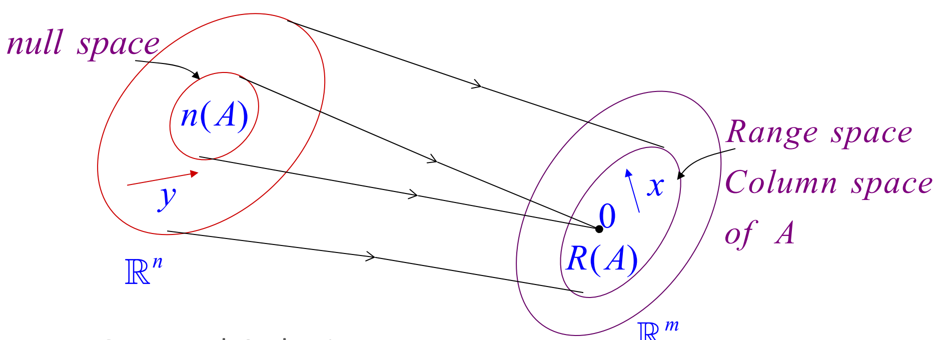
General Form

$$A_{(m\times n)}y_{(n\times 1)} = x_{(m\times 1)}$$



General Solution

$$y = A^{\#}x + \left[I_n - A^{\#}A\right]y_0$$

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Generalized Inverse

$$A^{\#}_{(n\times m)}: \quad AA^{\#}A = A$$

$$A_{(m\times n)} \;;\; rank(A) = r$$

Example $A = (2 \quad -1)$

$$A^{\#} = \begin{pmatrix} \frac{1}{2} + \frac{a}{2} \\ a \end{pmatrix}$$ For arbitrary a

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Example

$Ay = x$ with $A = (2 \quad -1)$ and $A^{\#} = \begin{pmatrix} \frac{1}{2} + \frac{a}{2} \\ a \end{pmatrix}$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A^{\#}x = \begin{pmatrix} \frac{1}{2} + \frac{a}{2} \\ a \end{pmatrix} x = \begin{pmatrix} \frac{1}{2}(1+a)x \\ ax \end{pmatrix}$$

$$Ay = AA^{\#} = (2 \quad -1)\begin{pmatrix} \frac{1}{2}(1+a)x \\ ax \end{pmatrix} = x$$

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Generalized Inverse

$$n > m$$
$$(r = m)$$

$$A_{(m\times n)}y_{(n\times 1)} = x_{(m\times 1)}$$

→ Less equations than unknowns

→ Free variables

→ ∞ solutions

Example

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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Generalized Inverse

$$n < m$$
$$(r = n)$$

→ More equations than unknowns

→ At most one solution

Example

$$\begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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Example

$$\begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

solution if x is in plan spanned by $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$

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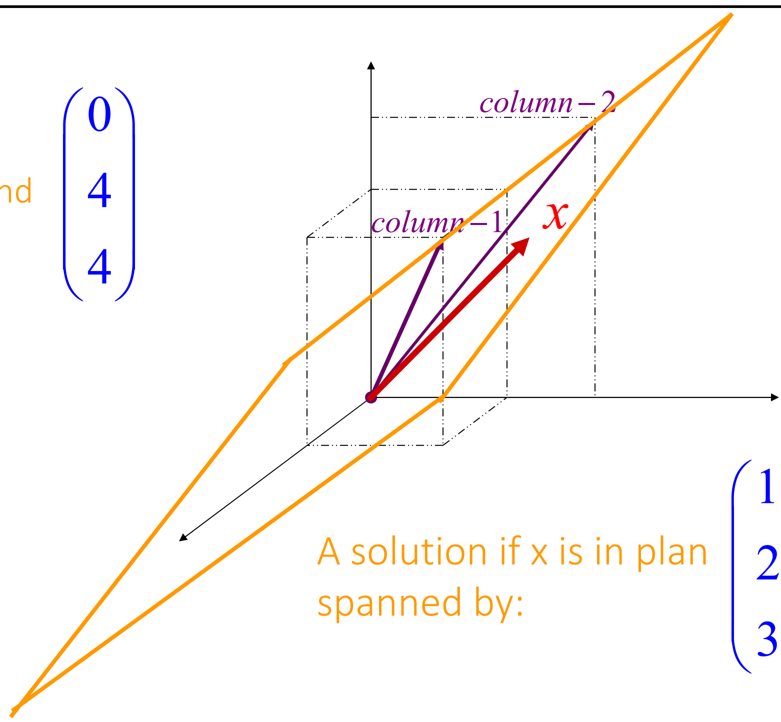
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$



A solution if x is in plan spanned by: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$

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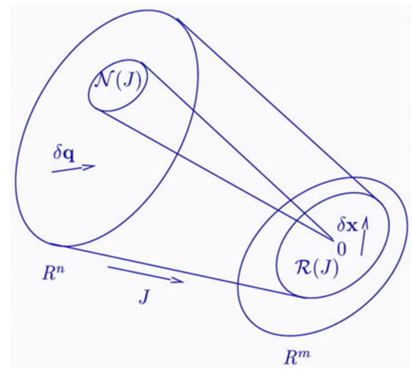
Jacobian Generalized Inverse

Generalized Inverse

$$J^{\#}: \quad J J^{\#} J = J$$



General Solution

$$\delta q = J^{\#} \delta x + N(J^{\#}) \delta q_0 \quad \text{with} \quad N(J^{\#}) = \left[I - J^{\#} J \right]$$

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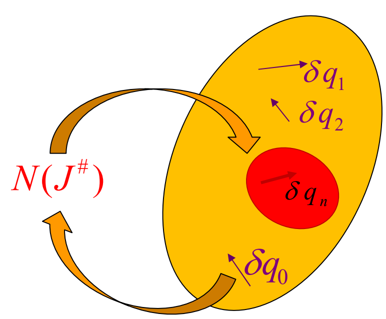
General Solution

$$\delta q = J^{\#} \delta x + \left[I - J^{\#} J \right] \delta q_0$$

Null Space (Kernel)

$$N(J^{\#}) = \left(I - J^{\#} J \right)$$

$$\delta q = J^{\#} \delta x + N(J^{\#}) \delta q_0$$



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Check

$$\delta q_n = \left(I - J^{\#} J\right) \delta q_0$$

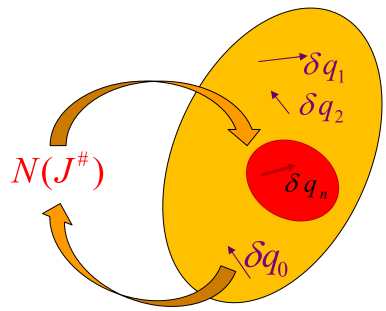
$$0 = J\, \delta q_n$$

$$\Rightarrow \quad 0 = J \left(I - J^{\#} J\right) \delta q_0$$

$$0 = J - J\, J^{\#} J$$

$$\Rightarrow$$

$$J^{\#} : \qquad J \triangleq J\, J^{\#} J$$



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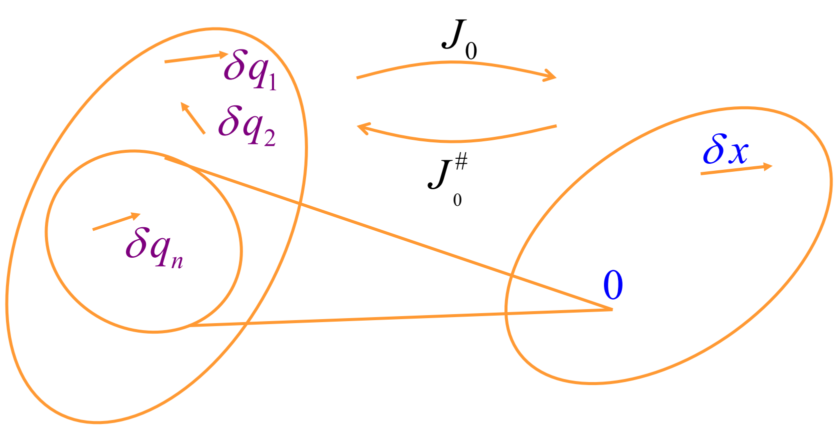
General Solution



$$J^{\#} : J = J\, J^{\#} J$$

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Pseudo Inverse

$$A A^+ A = A$$

$$A^+ A A^+ = A^+$$

$$\left(A^+ A\right)^T = A^+ A$$

$$\left(A A^+\right)^T = A A^+$$

$$A^+ : unique$$

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Pseudo Inverse

Left Inverse

$m > n$

$(r = n)$

$$A^+ = \left(A^T A\right)^{-1} A^T$$

$$A^+ A = I$$

$m = n = r$

$$A^+ = A^{-1}$$

$$A^+ A = A A^+ = I$$

Right Inverse

$m < n$

$(r = m)$

$$A^+ = A^T (A A^T)^{-1}$$

$$A A^+ = I$$

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Generalized Inverse

Left Inverse

$m < n$
$(r = m)$

$$A^{\#} = \left(A^T W^{-1} A\right)^{-1} A^T W^{-1}$$

$$A^{\#} A = I$$

$m = n = r$ $\quad A^{\#} = A^{-1} \quad A^{\#} A = A\,A^{\#} = I$

Right Inverse

$m < n$
$(r = m)$

$$A^{\#} = W^{-1} A^T (A W^{-1} A^T)^{-1}$$

$$A A^{\#} = I$$

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Reduction to the Basic Kinematic Model

Initial Problem (m equations)

$$J\,\delta q = \delta x$$

Reduced Problem (m_0 equations) $\quad J = E\,J_0$

$$\delta x = E(x)\,\delta x_0$$

$$J_0(q)\,\delta q = \delta x_0$$

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Solving $\delta x = E(x)\,\delta x_0$

$E(x) : m \times m_0 \; matrix \; (m \geq m_0)$

$- rank\left(E(x)\right) \leq m_0$

$- rank\left(E(x)\right) < m_0$

At Singular configuration
of the presentation

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Left Inverse

If $rank\left(E(x)\right) = m_0$ the system has a unique solution:

$$\delta x_0 = E^{+}_{(m_0 \times m)}(x)\;\delta x$$

E^{+} : is such that $E^{+}E = \mathrm{I}_{m_0}$

$$E^{+} = \left(E^{T}E\right)^{-1} E^{T}$$

and

$$E^{+}(X) = \begin{pmatrix} E^{+}_{p}(x_P) & 0 \\ 0 & E^{+}_{r}(x_R) \end{pmatrix}$$

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System

$$\delta x_{m\times 1} = E_{m\times m_0}\,\delta x_{0\,m_0\times 1}$$

$$E^T_{m_0\times m}\,\delta x_{m\times 1} = \left(E^T E\right)_{m_0\times m_0}\,\delta x_{0\,m_0\times 1}$$

$$\left(E^T E\right)^{-1} E^T \delta x = \delta x_0$$

$$\delta x_0 = E^+ \delta x$$

$$E^+ = \left(E^T E\right)^{-1} E^T$$

$$E^+ E = \underbrace{\left(E^T E\right)^{-1} E^T}_{\text{Left Inverse}} E = I$$

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Position Representations

Cartesian Coordinates (x, y, z)

$$E_P(x_P) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \;\; y \;\; z)^T = (\rho\cos\theta \;\; \rho\sin\theta \;\; z)^T$

$$E_P(x_P) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Position Representations

Cartesian Coordinates (x, y, z)

$$E_P^+(x_P) = E_P^{-1}(x_P) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

$$E_P^{-1}(x_P) = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ -\sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Spherical Coordinates (ρ, θ, ϕ)

Using

$$(x \;\; y \;\; z)^T = (\rho\cos\theta\sin\phi \;\;\; \rho\sin\theta\sin\phi \;\;\; \rho\cos\theta)^T$$

$$E_P(x_P) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & \_\sin\phi/\rho \end{pmatrix}$$

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Spherical Coordinates (ρ,θ,ϕ)

$$E_P^{-1}(x_P)=\begin{pmatrix}\cos\theta\sin\phi & \rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ -\sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi\end{pmatrix}$$

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Rotation Representations

Direction Cosines

$$x_r=\begin{bmatrix}r_1\\ r_2\\ r_3\end{bmatrix};\ E_r(x_r)=\begin{pmatrix}-\hat{r}_1\\ \hline -\hat{r}_2\\ \hline -\hat{r}_3\end{pmatrix}$$

$$E_r^+=\left(E_r^T E_r\right)^{-1}E_r^T$$

$$\left(E_r^T E_r\right)^{-1}=\left(\hat{r}_1^T\hat{r}_1+\hat{r}_2^T\hat{r}_2+\hat{r}_3^T\hat{r}_3\right)^{-1}$$

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$$E^T E = \sum \hat{r}_i^T \hat{r}_i = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2I_3$$

$$\forall x_r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \Rightarrow E^T E = \sum \hat{r}_i^T \hat{r}_i = 2I_3$$

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$$\left(E_r^T E_r\right)^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{-1} = \tfrac{1}{2} I_3$$

$$E_r^+ = \left(E_r^T E_r\right)^{-1} E_r^T$$

$$E_r^+ = \frac{1}{2} E_r^T$$

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Angular Velocity

$$x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}; \quad E_r(x_r) = \begin{pmatrix} -\hat{r}_1 \\ -\hat{r}_2 \\ -\hat{r}_3 \end{pmatrix}$$

$$\dot{x}_r = E_r \omega$$

Solution $\omega = \frac{1}{2} E_r^T \dot{x}_r$

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Direction Cosines – Rotation Error

Instantaneous Angular Error

$$x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}; \; x_{rd} = \begin{bmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{bmatrix}$$

$$\delta x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} - \begin{bmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{bmatrix}$$

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$$\omega = \frac{1}{2} E_r^T \dot{x}_r$$

$$\delta\phi = \frac{1}{2} E_r^T \delta x_r$$

$$\delta x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} - \begin{bmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{bmatrix}$$

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$$\delta x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} - \begin{bmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{bmatrix}$$

$$E_r^+ = \frac{1}{2} E_r^T$$

$$E_r^+ \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \frac{1}{2}(\hat{r}_1 r_1 + \hat{r}_2 r_2 + \hat{r}_3 r_3) \equiv 0$$

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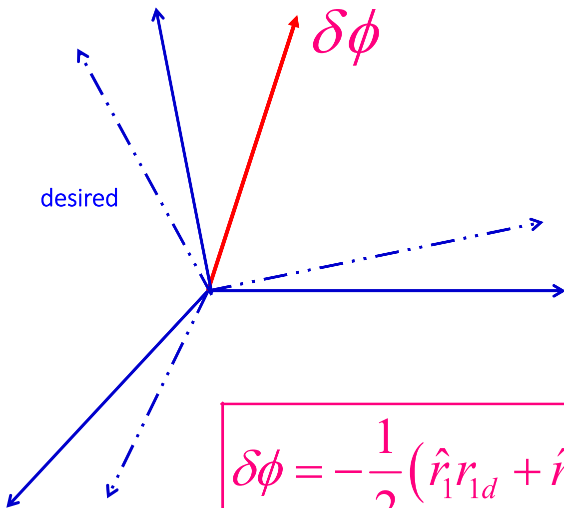
Instantaneous Angular Error



$$\delta\phi = -\frac{1}{2}\left(\hat{r}_1 r_{1d} + \hat{r}_2 r_{2d} + \hat{r}_3 r_{3d}\right)$$

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Euler Angles

$$E_r(x_r) = \begin{pmatrix} -{}^{S\varphi C\theta}/_{S\theta} & {}^{C\varphi C\theta}/_{S\theta} & 1 \\ C\varphi & S\varphi & 0 \\ {}^{S\varphi}/_{S\theta} & -{}^{C\varphi}/_{S\theta} & 0 \end{pmatrix}$$

$$E_r^{-1}\left(x_r\right) = \begin{pmatrix} 0 & \cos\psi & \sin\psi \sin\theta \\ 0 & \sin\psi & -\cos\psi \sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix}$$

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Euler Parameters

$$x_r = \lambda = \left(\lambda_0 \; \lambda_1 \; \lambda_2 \; \lambda_3\right)^T \qquad \check{\lambda} = \begin{pmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix}$$

$$\dot{\lambda} = \frac{1}{2} \check{\lambda} \, \omega$$

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Euler Parameters

Observing

$$\check{\lambda}^T \check{\lambda} = I_3$$

$$E_r^+\left(x_r\right) = 2 \begin{pmatrix} -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}$$

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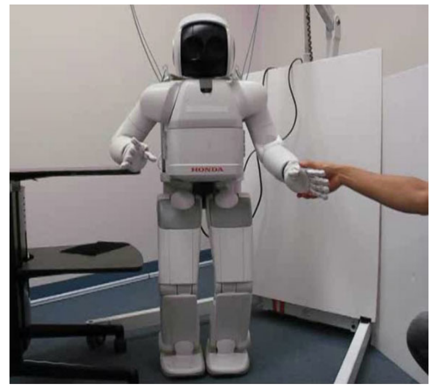
Redundancy: Inverse of the Basic Kinematic Model

System

$$\delta x_{0(m_0\times 1)} = J_0(q)_{(m_0\times n)}\delta q_{(n\times 1)}$$

General Solution

$$\delta q = J_0^{\#}\,\delta x_0 + \underbrace{\left[I_n - J_0^{\#}J_0\right]\delta q_0}_{\delta q_n}$$



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