Xqueens and Xqueenons

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Imagine an $n \times n$ board in which every cell ij, for $1 \le i, j \le n$, is divided by an '$\times$' into four isosceles right triangles, $(N_{ij}, E_{ij}, S_{ij}, W_{ij})$. Figure 1 illustrates the case $n = 5$:

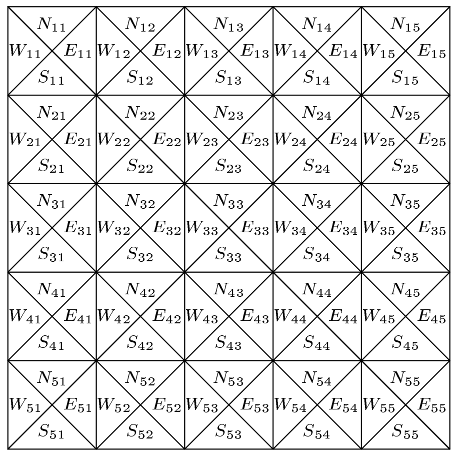


FIGURE 1.

This diagram was inspired by the concept of *queenons* in a recent breakthrough paper by Michael Simkin [2], who has determined the asymptotic number of solutions to the classical n queens problem; Figure 1 is equivalent to Figure 4 in [2]. (A similar board appeared in a 19th-century game called "Chat et Souris" [1, page 432], although tokens in that game were placed on the vertices, not in the triangles. The edges in this diagram constitute the graph of king moves, $P_n \boxtimes P_n$.)

The "n Xqueens problem" is to place $4n$ tokens on this board, satisfying the following ten conditions:

(N) Each row contains exactly one token in an N triangle.
(EW) Each row contains exactly two tokens in E or W triangles.
(S) Each row contains exactly one token in an S triangle.
(E) Each column contains exactly one token in an E triangle.
(NS) Each column contains exactly two tokens in N or S triangles.
(W) Each column contains exactly one token in a W triangle.
(NE) Each diagonal ('\') contains at most two tokens in N or E triangles.
(SW) Each diagonal contains at most two tokens in S or W triangles.
(NW) Each antidiagonal ('/') contains at most two tokens in N or W triangles.
(SE) Each antidiagonal contains at most two tokens in S or E triangles.

For example, we obtain a solution to the n Xqueens problem by putting independent solutions to the n queens problem into each of the N, E, S, and W triangles. But solutions can have much more variety: Indeed, the 2 queens problem can't be solved, but the 2 Xqueens problem has six solutions, exhibited in Figure 2.

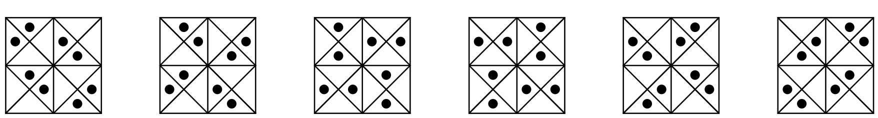


FIGURE 2.

Notice that tokens can "attack" each other, but only to a limited extent.

(What shall the tokens be called? I debated whether to call them "queens" or "quarterqueens" or "pawns" or "farthings." In this note I shall call them "Xqueens," even though the n Xqueens problem requires the placement of $4n$ Xqueens. There are n north Xqueens to be placed, as well as n east Xqueens, n south Xqueens, and n west Xqueens.)

There's an amusing way to generate all solutions to the n Xqueens problem using the DLX3 algorithm (Algorithm 7.2.2.1M), by analogy with the way the ordinary n queens problem can be solved with Algorithm 7.2.2.1X as an exact cover problem with n^2 options. (See 7.2.2.1–(23) in [3].) But for Xqueens we construct an MCC problem with $4n^2$ options, one for each triangle; again each option contains four items. This construction appears just after the references below, so as not to spoil the fun for readers who want to discover it for themselves.

It turns out that the total number of solutions, for $n = (1, 2, 3, 4, 5, 6)$, is exactly (1, 6, 132, 20742, 5834838, 4198336824). Here are three more-or-less random solutions for $n = 8$, of which there must be zillions:

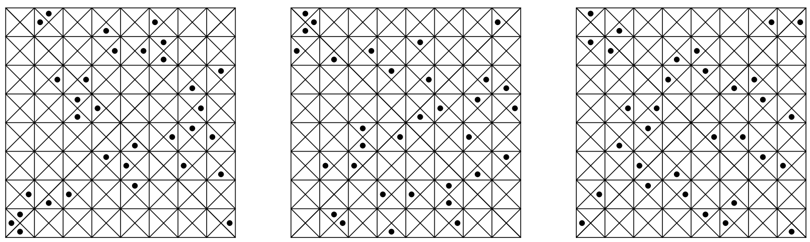


FIGURE 3.

The right-hand example is "sparse," in the sense that no cell contains more than one Xqueen. There are (0, 0, 0, 1320, 178584, 113260460, 146777600320) sparse solutions, for $n = (1, 2, 3, 4, 5, 6, 7)$. That example is also "paired," in the sense that each row contains either two Xqueens in the east or two in the west, and each column contains either two in the north or two in the south. (Thus no row mixes both east and west; no column mixes both north and south.) There are (0, 4, 0, 528, 0, 3938208, 0) paired solutions, for $n = (1, 2, 3, 4, 5, 6, 7)$. And the number of solutions that are *both* sparse and paired turns out to be (0, 0, 0, 140, 0, 317544, 0, 54472800228, 0), for n up to 9.

A puzzle for the reader: Is there a sparse, paired solution to the 8 Xqueens problem that also has (i) all north Xqueens in the four leftmost columns; (ii) all east Xqueens in the four topmost rows; and (iii) no Xqueens adjacent to either main diagonal? (See the answer at the end.)

Each of the solutions for $n = 2$ has four-fold symmetry; but most of the solutions, like the three above, lead to seven others if we rotate them and/or flip them over.

Closer study reveals, in fact, that there's another automorphism — that is, another transformation that takes solutions into solutions — which breaks the rules of ordinary geometry. The reader can check that the swaps

$$N_{ij} \leftrightarrow N_{\bar{\imath}j}, \quad E_{ij} \leftrightarrow W_{\bar{\imath}j}, \quad S_{ij} \leftrightarrow S_{\bar{\imath}j}, \quad W_{ij} \leftrightarrow E_{\bar{\imath}j}, \qquad \text{where } \bar{\imath} = n+1-i,$$

preserve all of the necessary conditions. This transformation also preserves sparsity and pairing. In fact — surprise — the middle solution in Figure 3 was obtained in this way from the left-hand solution.

(It's quite amazing that this transformation works. For example, when $n = 8$ it moves the corner triangles N_{11} and W_{11} into the positions of the *non*-corner triangles N_{81} and E_{81}!)

Armed with all of these transformations, the solutions fall into equivalence classes, with up to sixteen in each class. For example, each of the 132 solutions for $n = 3$ is equivalent to one of the following eleven:

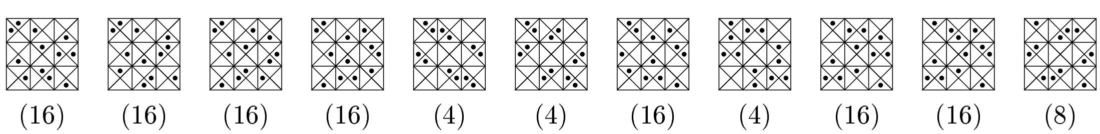


FIGURE 4.

Xqueenons. Solutions to the Xqueens problem are special cases of a much more general concept that Simkin [2] has called a *queenon*. A queenon is an assignment of a nonnegative real-valued density to the points of the unit square so that (i) the measure of any horizontal or vertical slice is *equal to* the width of that slice; and (ii) the measure of any 45°-degree diagonal slice is *less than or equal to* the width of that slice, as measured on the boundary. An "Xqueenon" of order n is a queenon that's constant on each of the $4n^2$ triangles of an X-decomposition. A "step queenon" is an Xqueenon whose north, east, south, and west densities agree on each of the n^2 cells.

(In these terms, a solution to the n Xqueens problem is an Xqueenon in which every triangle has density 0 or n. A solution to the n queens problem is a step queenon in which every cell has density 0 or n.)

Let the densities of an Xqueenon's triangles be n_{ij}, e_{ij}, s_{ij}, w_{ij} for $1 \le i, j \le n$. It's not difficult to deduce that the allowable densities are characterized by linear equalities and inequalities:

$$\Sigma N_i = n, \quad \Sigma E_j = n, \quad \Sigma S_i = n, \quad \Sigma W_j = n,$$
$$\Sigma NS_i = 2n, \quad \Sigma EW_j = 2n, \quad \Sigma NW_k \le 2n, \quad \Sigma SE_k \le 2n, \quad \Sigma NE_k \le 2n \quad \Sigma SW_k \le 2n,$$

where $1 \le i, j \le n$ and $0 \le k \le 2n$ and

$$\Sigma N_i = \sum_{j=1}^{n} n_{ij}; \quad \Sigma E_j = \sum_{i=1}^{n} e_{ij}; \quad \Sigma S_i = \sum_{j=1}^{n} s_{ij}; \quad \Sigma W_j = \sum_{i=1}^{n} w_{ij};$$
$$\Sigma NS_j = \sum_{i=1}^{n} (n_{ij} + s_{ij}); \quad \Sigma EW_i = \sum_{j=1}^{n} (e_{ij} + w_{ij});$$

$$\begin{cases} \begin{cases} \Sigma NW_k = \sum_{i=1}^{k} \big(n_{i(k+1-i)} + w_{i(k+1-i)}\big), \ \Sigma SE_k = \sum_{i=1}^{k} \big(s_{i(k+1-i)} + e_{i(k+1-i)}\big), \\ \Sigma NE_k = \sum_{i=1}^{k} \big(n_{i(i+n-k)} + e_{i(i+n-k)}\big), \quad \Sigma SW_k = \sum_{i=1}^{k} \big(s_{i(i+n-k)} + w_{i(i+n-k)}\big), \end{cases} & \text{if } 0 \le k \le n; \\ \begin{cases} \Sigma NW_k = \sum_{i=k+1-n}^{n} \big(n_{i(k+1-i)} + w_{i(k+1-i)}\big), \quad \Sigma SE_k = \sum_{i=k+1-n}^{n} \big(s_{i(k+1-i)} + e_{i(k+1-i)}\big), \\ \Sigma NE_k = \sum_{i=k+1-n}^{n} \big(n_{i(i+n-k)} + e_{i(i+n-k)}\big), \quad \Sigma SW_k = \sum_{i=k+1-n}^{n} \big(s_{i(i+n-k)} + w_{i(i+n-k)}\big), \end{cases} & \text{if } n \le k \le 2n. \end{cases}$$

The simplest solution to all these constraints arises when all the densities are uniform: $n_{ij} = e_{ij} = s_{ij} = w_{ij} = 1$ for all i and j.

Let $g(0) = 0$ and $g(x) = x \ln x$ for $x > 0$. Simkin [2] defined the "Q-entropy" of a queenon in a way that boils down to the following formula, in the case of an Xqueenon:

$$H_q = -KL_0 - KL_+ - KL_- - 3,$$

where

$$KL_0 = \frac{1}{4n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \big(g(n_{ij}) + g(e_{ij}) + g(s_{ij}) + g(w_{ij})\big);$$
$$KL_+ = \sum_{k=1}^{2n} \frac{1}{n} \int_0^1 g\Big(1 - \frac{1}{2n}\big((1-y)\Sigma SE_{k-1} + y\Sigma NW_k\big)\Big)\,dy;$$
$$KL_- = \sum_{k=1}^{2n} \frac{1}{n} \int_0^1 g\Big(1 - \frac{1}{2n}\big((1-y)\Sigma SW_{k-1} + y\Sigma NE_k\big)\Big)\,dy.$$

For example, when all the densities are 1, we have $\Sigma NW_k = \Sigma SE_k = \Sigma NE_k = \Sigma SW_k = 2\min(k, 2n-k)$; hence $KL_0 = 0$ and

$$\begin{aligned} KL_+ = KL_- &= 2\sum_{k=1}^{n} \frac{1}{n} \int_0^1 g\Big(1 - \Big(\frac{k-1}{n} + \frac{ky}{n}\Big)\Big)\,dy \\ &= 2\sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} g(1-x)\,dx = 2\int_0^1 g(1-x)\,dx = -2/4. \end{aligned}$$

It follows that the Q-entropy of the uniform queenon is -2.

Simkin proved that there's a *unique* queenon γ^* whose Q-entropy achieves the supremum of H_q over all queenons, and that the Q-entropy of this champion queenon lies between -1.95 and -1.94. Furthermore, if we look at the step queenons of all solutions to the n queens problem, their average density approaches the density of γ^*, as $n \to \infty$. And the same limiting distribution applies to the average density of all solutions to the n Xqueens problem.

Let's look therefore at the 4198336824 solutions to the 6 Xqueens problem. Their average density involves just nine numbers, namely n_{ij} for $1 \le i, j \le 3$, because of symmetry. Indeed, it's easy to see that $n_{ij} = n_{i(7-j)} = n_{(7-i)j} = n_{(7-i)(7-j)}$, and that $s_{ij} = n_{ij}$, $e_{ij} = w_{ij} = n_{ji}$. The actual numbers are

$$\frac{6}{4198336824}\begin{pmatrix} 462295765 & 737601579 & 899271068 \\ 738880358 & 706369307 & 653918747 \\ 897992289 & 655197526 & 545978597 \end{pmatrix} \approx \begin{pmatrix} 0.66068 & 1.05413 & 1.28518 \\ 1.05596 & 1.00950 & 0.93454 \\ 1.28335 & 0.93637 & 0.78028 \end{pmatrix}.$$

Surprisingly, we don't have $n_{ij} = n_{ji}$, because of the strange automorphism discussed earlier. (That automorphism takes Xqueen solutions into Xqueen solutions, but it does *not* preserve Q-entropy.) The densities are very close, however; so the associated Xqueenon is almost indistinguishable from a step queenon.

The Q-entropy of this particular Xqueenon turns out to be -1.94861. It is *not* the optimum Xqueenon for $n = 6$, despite the fact that's the average of all solutions. The optimum one has 18 distinct densities, namely n_{ij} for $1 \le i \le n$ and $1 \le j \le \lceil n/2 \rceil$, because the strange automorphism doesn't apply. We have $n_{ij} = n_{i(7-j)}$; also $n_{ij} = e_{j(7-i)} = s_{(7-i)(7-j)} = w_{(7-j)i}$. The approximate optimum values of n_{ij} are

$$\begin{pmatrix} 0.65670 & 1.04263 & 1.30068 \\ 0.99222 & 1.00314 & 1.00464 \\ 1.21536 & 0.97510 & 0.80955 \\ 1.23891 & 0.96708 & 0.79401 \\ 1.09084 & 0.97298 & 0.93617 \\ 0.80598 & 1.03907 & 1.15495 \end{pmatrix}$$

and the associated Q-entropy is -1.94584.

Incidentally, I hoped at one time to prove that every Xqueenon is a convex combination of the Xqueenons that arise from solutions to the n Xqueens problem. That conjecture turned out to be false already for $n = 3$, because of the Xqueenon with $n_{11} = 3/2$, $n_{12} = 0$, $n_{21} = n_{22} = 1$, $n_{31} = 1/2$, $n_{32} = 2$, and with all other densities determined by 8-fold symmetry. This one can't be a convex combination of the 132 solutions, because the eleven equivalence classes shown earlier have no solution with $n_{12} = n_{21} = n_{23} = n_{32} = 0$.

References.

[1] Jean-Marie Lhôte, *Histoire des jeux de société* (Paris: Flammarion, 1994), 672 pages.

[2] Michael Simkin, "The number of n-queens configurations," arXiv:2107.13460 [math.CO], 51 pages.

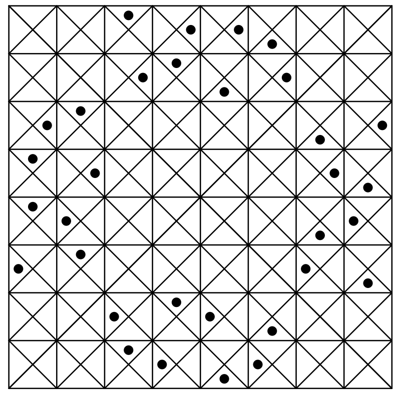
[3] Donald E. Knuth, *The Art of Computer Programming*, Volume 4, Fascicle 5 (Boston, Massachusetts: Addison–Wesley, 2019), viii + 384 pages.

The MCC construction promised above. Let there be primary items nk, ek, sk, wk, ns$k$, and ew$k$ for $0 \le k < n$, together with primary items ne$k$, nw$k$, se$k$, and sw$k$ for $0 \le k < 2n-1$. The n, e, s, and w items have multiplicity 1; the ns and ew items have multiplicity 2; and the multiplicities of the ne, nw, se, and sw items are the interval $[0\,..\,2]$. The options, for $0 \le i, j < n$, are 'ni ns$j$ ne$(i+n-1-j)$ nw$(i+j)$'; 'ew$i$ ej ne$(i+n-1-j)$ se$(i+j)$'; 'si ns$j$ sw$(i+n-1-j)$ se$(i+j)$'; 'ew$i$ wi sw$(i+n-1-j)$ nw$(i+j)$'. For example, when $n = 8$ the four options for $(i, j) = (2, 5)$ are

'n2 ns5 ne4 nw7'; 'ew2 e5 ne4 se7'; 's2 ns5 sw4 se7'; 'ew2 w5 sw4 nw7'.

To get only sparse solutions, introduce secondary items xij for $0 \le i, j < n$, and insert xij into each option. To get only paired solutions, introduce secondary items rk and ck for $0 \le k < n$; insert ci:n, rj:e, ci:s, or rj:w into the north, east, south, or west options, respectively.

Answer to the puzzle. There are 16 ways to do it, all variants of the following:



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