☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ ERRATA TO VOLUME 2 (after 2021) ☆☆☆☆

This document is a transcript of the notes that I have been making in my personal copy of *The Art of Computer Programming*, Volume 2 (third edition, 45th printing) since it was first printed in 2022. Previous errata are recorded in another file '`all2-pre.ps`'.

Three levels of updates — "errors," "amendments," and "plans" — appear, indicated by three different typographic conventions:

▸**Page 666** line 1 ——— 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \mathrel{\not\rightarrow} n+1$

Page 666 line 2 ——— 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . . ' leading to the date on which I recorded the plan in my files.

My personal file of updates also includes a fourth category of items, not shown in this list. They are miscellaneous minor corrections or improvements that most readers don't want to know about, because they are so trivial. If you really want to see all of the gory details, you can download the full list from Internet webpage

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

by selecting the "long form" of the errata.

My shelves at home are bursting with preprints and reprints of significant research results that I want to digest and summarize, where appropriate, in the ultimate edition of Volume 2. I didn't do that in the third edition because I would surely have to do it over again later: New results continue to pour forth at a great rate, and I will have time to rewrite that volume only once. Volumes 4 and 5 need to be finished first. So I've put most of my effort so far into writing up those parts of the total picture that seem to have converged to their near-final

form. It follows, somewhat paradoxically, that the updates in this document are most current in the areas where there has been least activity.

On the other hand I do believe that the changes listed here bring Volume 2 completely up to date in two respects: (1) All of the research problems in the previous edition — i.e., all exercises that were rated 46 and above — have received new ratings of 45 or less whenever I learned of a solution; and in such cases, the answer now refers to that solution. (2) All of the historical information about pioneering developments has been amended whenever new details have come to my attention.

The ultimate, glorious, 100% perfect editions of Volumes 1–4 are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 1B, Stanford University, Stanford CA 94305-9015, or by email to `taocp@cs.stanford.edu`. *(Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although I'm working full time on Volume 4 these days, I will try to reply to all such messages within a year of receipt. Current news about The Art of Computer Programming is posted on*

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

and updated regularly.

— Don Knuth, January 2011

Oh! If only someone would give me time, time,
time to do everything properly,
to read everything at my own *tempo,*
to take it apart and put it together again.

— KARL BARTH (1922)

☆☆☆☆☆ SEMINUMERICAL ALGORITHMS ☆☆☆☆☆

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Last updated 13 November 2023

Most of these corrections have already been made in recent printings.

▸**Page 57** Smirnov reference near the bottom — 10 Nov 2022

176–206 ↜ 179–206

Page 61 line 7 — 14 Dec 2022

eleven kinds ↜ twelve kinds

Page 198 line 22 — 08 Nov 2023

1586–1610. ↜ 1586–1610. (In India, al-Kāshī's contemporary Mādhava had computed $9 \cdot 10^{11}\pi \approx 2827433388233$, according to Nīlakantha's commentary on Gaṇita §10 of the *Āryabhaṭīya*.)

Page 199 line 9 — 08 Nov 2023

man's custom ↜ human custom

Page 200 lines 13–14 — 10 Aug 2022

as early as 1679 . . . "decimal"). ↜ already in 1679 or earlier, and that he often experimented with radix-16 numbers.

Page 200 line 17 — 10 Aug 2022

1973). Zacher points out ↜ 1973), and extended by L. Strickland and H. Lewis (Cambridge, Mass.: MIT Press, 2022), who have provided English translations. Zacher pointed out

▸**Page 200** line −18 — 10 Aug 2022

35-digit ↜ 35-place

Page 201 line 1 — 10 Aug 2022

More than a century later, a prominent ↜

The eccentric British schoolmaster T. W. Hill spent more than 50 years of his life experimenting with radix 16. His public lecture in 1845, published posthumously, described ingenious ways to say and to write hexadecimal numbers and fractions [*Selections from the Papers of the Late Thomas Wright Hill, Esq. F.R.A.S.* (London: 1860), 63–85]. But he gave no examples of arithmetic.

Independently, a prominent

Page 201 delete an obsolete paragraph — 10 Aug 2022

The book *History of Binary* . . . 243]. ↜

Page 202 line 5 — 08 Aug 2022

sexadecimal ↜ sexdecimal

Page 209 lines 25 and 26 — 02 Nov 2022

Several questions . . . by W. Parry, ↜ Intriguing questions concerning *irrational* radices have been investigated by A. Rényi, *Acta Math. Acad. Sci. Hung.* **8** (1957), 477–493; W. Parry,

Page 209 just before the exercises — 02 Nov 2022

Gray code ↜ regular continued fractions (Section 4.5.3); Gray codes

Page 213 line 2 — 18 Feb 2023

Crelle **127** (1904), 51–84. ↜
Jahresbreicht der Deutschen Mathematiker-Vereinigung **6** (1897), 83–88.

▸**Page 311** in the Harvey / van der Hoeven reference — 10 Nov 2022

191 ↜ (2) **193**

Page 409 lines −17 and −16 — 08 Sep 2023

all exponents . . . ; his ↜ all exponents $< 65{,}610{,}011$ were checked as of November 8, 2023; his

▸**Page 431** in Table 1 — 21 Nov 2022

rows C_1 and C_0: $d_2^2b_6^4/c_4^5$ ↜ $d_2^3b_6^4/c_4^5$
row D_0: $e_2^2c_4^2/d_2^3b_6^2$ ↜ $e_1^2c_4^2/d_2^3b_6^2$

▸**Page 432** three lines before (23) — 08 Dec 2022

$C_{n_2+n_j-1}$ becomes $C_{n_2-n_j-1}$

Page 477 lines 20 and 21 — 12 Aug 2022

using a backtrack . . . cases. ↜ using the backtrack method in a program called `ACHAINO` now on the Internet, involved a detailed examination of more than 100 cases.

Page 477 replacement for the table at the bottom — 13 Aug 2022

r	$c(r)$	r	$c(r)$	r	$c(r)$	r	$c(r)$
0	1	12	379	24	357887	36	550040063
1	2	13	607	25	685951	37	994660991
2	3	14	1087	26	1176431	38	1886023151
3	5	15	1903	27	2211837	39	3502562143
4	7	16	3583	28	4169527	40	6490123999
5	11	17	6271	29	7624319	41	11889505663
6	19	18	11231	30	14143037	42	22899028607
7	29	19	18287	31	25450463	43	41866170239
8	47	20	34303	32	46444543	44	76086635263
9	71	21	65131	33	89209343	45	142771387391
10	127	22	110591	34	155691199	46	257661019487
11	191	23	196591	35	298695487	47	498691112447

Page 478 lines 4–8 — 13 Aug 2022

The values listed ... this agrees ↜ The values of $c(r)$ listed here for $18 < r \le 28$ were discovered by Achim Flammenkamp before 1998, except that $c(24)$ was first computed by Daniel Bleichenbacher; $c(29)$ through $c(47)$ are due to Neill Clift, between 2004 and 2021. Notice that the formula $1 + r/(1 + 1/\lg r)$ gives a fairly good approximation to $\lg c(r)$ for $10 < r \le 47$; this agrees

Page 478 three lines before (49) — 13 Aug 2022

large r. ↜ large r. (Empirically, $d(r) \approx 1.7c(r)$ for $35 \le r \le 44$.)

Page 478 lines −11 through −5 — 12 Aug 2022

Computer calculations ... Arnold Scholz ↜
Notice that 2^n-1 is the worst case for the binary method, because $\nu(2^n-1) = n$. E. G. Thurber [*Discrete Math.* **16** (1976), 279–289] has shown that several of these values, including the case $n = 32$, can actually be calculated by hand. Extensive computer calculations by Neill Clift [*Computing* **91** (2011), 265–284; also Thurber and Clift, *Discrete Math.* **344** (2021), #112200, 15 pp.] show that $l(2^n - 1)$ is in fact exactly *equal* to $n - 1 + l(n)$ whenever $l(n) \le 9$.
Arnold Scholz

Page 485 rewording exercise 37 — 14 Aug 2022

contains m given ... (25). ↜ contains each of the numbers $n_1, \ldots, n_m$. Generalizing (25), prove that $l(n_1, \ldots, n_m) \le \lambda(n) + m\lambda(n)/\lambda\lambda(n) + O(\lambda(n)\lambda\lambda\lambda(n)/\lambda\lambda(n)^2)$, $n = \max\{n_1, \ldots, n_m\}$.

▸**Page 645** in the Morris reference, beginning of answer 34 — 10 Nov 2022

73–145 ↜ 73–143

Page 667 in the last paragraph of answer 41 — 29 Dec 2022

Further refinements ... 768. ↜ Further refinements due to numerous other researchers have made it possible to go much higher; the current record, due to David Baugh and Kim Walisch in 2022, is $\pi(10^{29}) = 152069810971427216609425806 3$. (See OEIS sequence A006880.)

Page 667 at the end of answer 41 — 30 Dec 2022

large.] ↜ large. See also the elementary method of D. Hirsch, I. Kessler, and U. Mendlovic, arXiv:2212.09857 [math.NT] (2022), 60 pages.]

▸**Page 673** bottom line — 08 Dec 2022

$\pm\ell_k^{n_{k-1}} / \prod_{1<j<k} \ell_j^{\delta_{j-1}(\delta_j-1)}$. ↜ $\pm\prod_{j=2}^{k} \ell_j^{e_j}$, where $e_j = \delta_{j-1}\prod_{i=j}^{k-1}(1-\delta_i)$, $\delta_i = \deg(u_i) - \deg(u_{i+1})$.

Page 697 in answer 39 — 14 Aug 2022

line 8: 21.] ↜ 21.] In this argument $\{n_1, n_2, \ldots, n_m\}$ can be any *multiset*.
line 16: 2–3 ↜ 2–4

Page 698 line 2 — 22 Feb 2022

for any $(r + 1) \times (s + 1)$ ↜ for any nonzero $(r + 1) \times (s + 1)$

Page 698 bottom line of answer 41 — 22 Feb 2022

for $\{n_1, \ldots, n_m\}$ for odd numbers $\{n_1, \ldots, n_m\}$

Page 700 near the end of answer 12 — 10 Nov 2022

arXiv:1211.6320 [cs.CC] (2013), 10 pages. ↜ *Linear Algebra and Its Applications* **438** (2013), 4500–4509.

Page 715 the end of answer 62 — 10 Nov 2023

235. ↜ 235. For a modern treatment, including proofs that $\underline{\text{rank}}(T(2,2,3)) = 10$ and $\underline{\text{rank}}(T(2,3,3)) = 14$, see A.Conner, A. Harper, and J. M. Landsberg, *Forum of Mathematics, Pi* **11** (2023), e17:1–30.

Page 715 last line of answer 64 — 10 Nov 2023

The border rank of $T(2,2,2)$ remains unknown.] ↜ The paper cited in answer 62 proves that $\underline{\text{rank}}(T(2,2,2)) = 7$ and $\underline{\text{rank}}(T(3,3,3)) \geq 17$.]

Page 737 and following — 01 Feb 2022

Miscellaneous changes to the existing index of Volume 2 are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.

al-Samaw'al (= as-Samaw'al), ibn Yaḥyā ibn Yahūda al-Maghribī (ابن يحيى بن يهودا المغربي السموءل, modern form السموأل), 198.
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