☆ THE ART OF COMPUTER PROGRAMMING ☆

☆☆☆☆ ERRATA TO VOLUME 1 (after 2021) ☆☆☆☆

This document is a transcript of the notes that I have been making in my personal copy of *The Art of Computer Programming*, Volume 1 (third edition, 27th printing), since it was first printed in 2022. Previous errata are recorded in another file '`all1-pre.ps`'.

Three levels of updates — "errors," "amendments," and "plans" — appear, indicated by three different typographic conventions:

▸**Page 666** line 1 ——— 04 Jul 1776

Technical or typographical errors (aka bugs) are the most critical items, so they are flagged with a '▸' preceding the page number. The date on which I first was told about the bug is shown; this is the effective date on which I paid the finder's fee. The necessary corrections are indicated in a straightforward way. If, for example, the book says 'n' where it should have said '$n+1$', the change is shown thus:

$n \mathrel{\not\rightarrow} n+1$

Page 666 line 2 ——— 14 Jul 1789

Amendments to the text appear in the same format as bugs, but without the '▸'. These are things I wish I had known about or thought of when I wrote the original text, so I added them later. The date is the date I drafted the new text.

Page 666 line 3 . . . . . 20 Nov 1917

Plans for the future represent a third kind of item. In such notes I sketched my intentions about things that I wasn't ready to flesh out further when I wrote them down. You can identify these items because they're written in slanted type, and preceded by a bunch of dots '. . . . . ' leading to the date on which I recorded the plan in my files.

My personal file of updates also includes a fourth category of items, not shown in this list. They are miscellaneous minor corrections or improvements that most readers don't want to know about, because they are so trivial. If you really want to see all of the gory details, you can download the full list from Internet webpage

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

by selecting the "long form" of the errata.

My shelves at home are bursting with preprints and reprints of significant research results that I want to digest and summarize, where appropriate, in the ultimate edition of Volume 1. I didn't do that in the third edition because I would surely have to do it over again later: New results continue to pour forth at a great rate, and I will have time to rewrite that volume only once. Volumes 4 and 5 need to be finished first. So I've put most of my effort so far into writing up those parts of the total picture that seem to have converged to their near-final

form. It follows, somewhat paradoxically, that the updates in this document are most current in the areas where there has been least activity.

On the other hand I do believe that the changes listed here bring Volume 1 completely up to date in two respects: (1) All of the research problems in the previous edition — i.e., all exercises that were rated 46 and above — have received new ratings of 45 or less whenever I learned of a solution; and in such cases, the answer now refers to that solution. (2) All of the historical information about pioneering developments has been amended whenever new details have come to my attention.

The ultimate, glorious, 100% perfect editions of Volumes 1–4 are works in progress. Please let me know of any improvements that you think I ought to make. Send your comments either by snail mail to D. E. Knuth, Computer Science, Gates Building 1B, Stanford University, Stanford CA 94305-9015, or by email to `taocp@cs.stanford.edu`. *(Use email for book suggestions only, please — all other correspondence is returned unread to the sender, or discarded, because I have no time to read ordinary email.) Although I'm working full time on Volume 4 these days, I will try to reply to all such messages within a year of receipt. Current news about The Art of Computer Programming is posted on*

`http://www-cs-faculty.stanford.edu/~knuth/taocp.html`

and updated regularly.

— Don Knuth, January 2011

What happened?
The subject took the bit in its teeth and ran away with it,
that's what happened.
I know now how Sir James Frazer felt when,
after setting out to dash off a brief monograph
on a single obscure rite, he found himself
in the embarrassing possession of
the 12 volumes of "The Golden Bough."

— WAVERLEY ROOT (1974)

No matter how many times you proofread your book,
after publication there will be roughly one typo per page.

— JOSEPH H. SILVERMAN (2017)

☆☆☆☆☆ FUNDAMENTAL ALGORITHMS ☆☆☆☆☆

Copyright © 2022, 2023, 2024 Addison–Wesley
Last updated 8 February 2024

Most of these corrections have already been made in recent printings.

Page 12 replacement for line −14 — 29 Mar 2023

$$p(0) = 1, \quad p(1) = 1, \quad p(2) = 2, \quad p(3) = 3, \quad p(4) = 5, \quad p(5) = 7.$$

Page 13 add a sentence following line −7 — 05 Dec 2023

legitimate. ↯ legitimate. [Actually $P(0)$ does happen to be true, in this particular context; but we weren't asked to prove it. So we didn't.]

Page 17 lines 20 and 21 — 17 Nov 2023

The first European ... 1575. ↯ Noteworthy early examples of inductive reasoning can be found in *Sefer Maasei Hoshev* by Levi ben Gershon in France (1321), and in *Arithmeticorum libri duo* by Francesco Maurolico in Italy (1557).

Page 21 line 2 after (1) — 07 Feb 2024

end with infinitely many 9s. ↯ end with infinitely many 9s. (Instead of $999\ldots$, we can use $000\ldots$ and add 1 to the digit at the left.)

Page 22 top lines — 07 Feb 2024

Throughout this section ... It is easy to prove ↯

Throughout this section, let the letter b stand for a positive real number. If n is an integer, then b^n and 0^n and $(-b)^n$ are defined by the familiar rules

$$b^0 = 1, \qquad b^n = b^{n-1}b \quad \text{if} \quad n > 0, \qquad b^n = b^{n+1}/b \quad \text{if} \quad n < 0; \tag{4}$$

$$0^0 = 1, \qquad 0^n = 0 \quad \text{if } n > 0; \qquad (-b)^n = (-1)^n b^n. \tag{5}$$

It is easy to prove

Page 22 through 26 — 07 Feb 2024

[renumber equations formerly numbered (5) through (19) to be (6) through (20)]

Page 27 line 5 — 01 Mar 2022

following equivalent ↯ following essentially equivalent

Page 27 line 11 after (2) — 07 Feb 2024

silly to interpret it as a sum on values of $n \geq j$; but ↯ silly to interpret it as a sum $a_j + a_j + a_j + \cdots$ over all indices n that satisfy $n \geq j$! But

Page 34 in exercise 8 — 24 Nov 2022

infinite series in which ↯ doubly infinite sequence $\langle a_{ij} \rangle$ of integers for which

Page 57 six lines after (13) — 11 Aug 2022

since the terms in Eq. (13) are zero when $k < 0$ or $k > r$. ↯ since we consider $\binom{r}{k}$ to be "strongly zero" when $k < 0$ or $k > r = \lfloor r \rfloor$ —strong enough to cancel negative powers of 0.

▸**Page 60** line −1 — 19 Jan 2023

assuming that $k \geq 0$ ↝ because $k \geq 0$

▸**Page 63** line −10 — 11 Aug 2022

$\sum_{k}$ ↝ $\sum_{k\geq 0}$

▸**Page 63** lines −3 and −2 — 11 Aug 2022

$\sum_{n}$ ↝ $\sum_{n\geq 0}$

▸**Page 70** append a new line to exercise 25 — 09 Aug 2022

where again the left-hand side is considered to be a polynomial in r.
[Also change '.' to ',' on the preceding line.]

Page 77 the line before (10) — 01 Mar 2022

$S_1 = x$ ↝ $S_0 = 0$

Page 77 line −7 — 01 Mar 2022

If $x > 0$, then ↝ *If $x > 0$ and $n \geq 0$, then*

▸**Page 79** line −3 — 29 Mar 2023

285 ↝ 284

▸**Page 80** second full paragraph, last line — 29 Mar 2023

126 ↝ 124

▸**Page 80** third full paragraph, line 4 — 06 Apr 2022

not greater than F_k, ↝ not greater than F_k, and $k > 1$,

Page 83 replacement for (13) — 07 Aug 2022

$$\hat{\phi} = 1 - \phi = -1/\phi = \tfrac{1}{2}(1 - \sqrt{5}). \tag{13}$$

▸**Page 85** in exercises 22 and 23 — 06 Apr 2022

number. ↝ number when $n \geq 0$.
[And exercise 25 should also specify that $n \geq 0$.]

▸**Page 86** in exercise 31 — 06 Apr 2022

ϕ^{-2n-1}. ↝ ϕ^{-2n-1}, when $n \geq 0$.

Page 88 line −10 — 09 Aug 2022

simplest example ↝ simplest nontrivial example

Page 92 replacement for (33) — 14 Jan 2024

$$h_m = \sum_{1\leq j_1\leq\cdots\leq j_m\leq n} x_{j_1}\ \ldots\ x_{j_m}; \qquad h_0 = 1. \tag{33}$$

▸**Page 93** replacement for (39) — 12 Mar 2023

$$h_m = \frac{1}{m}(S_1 h_{m-1} + S_2 h_{m-2} + \cdots + S_m h_0), \qquad m \geq 1. \tag{39}$$

▸**Page 93** line −3 — 29 Mar 2023

picture writing ↬ picture-writing

Page 94 replacement for second line of exercise 10 — 14 Jan 2024

$$e_m = \sum_{1\le j_1<\cdots<j_m\le n} x_{j_1}\ldots x_{j_m}; \qquad e_0 = 1.$$

Page 94 append new sentence to end of exercise 13 — 01 Jan 2024

(Assume that the real part of s is positive, so that the integral exists.)

Page 102 line 1 — 04 Jan 2024

$G_1(z) = q + pz$ ↬ $G_0(z) = 1$

Page 103 line 1 — 04 Jan 2024

replace z by $z+1$ ↬ replace z by $1+z$

Page 104 line 1 — 04 Jan 2024

zero except ↬ zero except possibly

Page 104 line 11 — 04 Jan 2024

random integer ↬ random nonnegative integer

▸**Page 105** bottom line — 29 Mar 2023

Reading, Mass. ↬ Cambridge, Mass.

Page 106 last line of exercise 21 — 25 Apr 2023

when $\epsilon \ge 0$, and obtain ↬ when $q = 1-p$ and $\epsilon \ge 0$. Also obtain

Page 107 line 2 of exercise 22 — 21 May 2023

$p_k + q_k = 1$ ↬ $p_k + q_k = 1$ and $0 \le p_k \le 1$

▸**Page 118** line −5 — 29 Mar 2023

1.2.11.3–(13) ↬ 1.2.11.1–(13)

Page 144 line 15 or 16 — 03 Nov 2022

always greater than 100 ↬ always 0100 or more

▸**Page 148** line *05* of Program P — 28 Aug 2022

```
EQU -1 ↬ EQU 99
```

▸**Page 182** line 3 — 29 Mar 2023

1957 ↬ 1967

▸**Page 183** line −4 — 29 Mar 2023

for all n ↬ for all $n \ge 1$

Page 185 new exercise — 24 Mar 2023

▸**38.** [*M22*] (*Conjugate permutations.*) When π and τ are arbitrary permutations, the permutation

$$\pi^\tau = \tau^- \pi \tau$$

is called "the conjugate of π by τ."

a) Prove that $\pi = \pi^\tau$ if and only if π commutes with τ.
b) Suppose π has the cycle form $(a_{11} \dots a_{1l_1}) \dots (a_{k1} \dots a_{kl_k})$. Prove that π^τ has the cycle form $(b_{11} \dots b_{1l_1}) \dots (b_{k1} \dots b_{kl_k})$, where τ takes $a_{ij} \mapsto b_{ij}$.
c) Find all τ for which $((1\,2\,3)(4\,5)(6\,7\,8))^\tau = (8\,7\,6)(5\,4)(3\,2\,1)$.
d) Find all τ for which $((1\,2\,3)(4\,5)(6\,7\,8))^\tau = (8\,7\,6\,5)(4\,3\,2\,1)$.
e) True or false: Every permutation π is conjugate to its inverse, π^-.
f) True or false: $(\pi\sigma)^\tau = \pi^\tau \sigma^\tau$ and $\pi^{(\sigma\tau)} = (\pi^\sigma)^\tau$, for all π, σ, and τ.

▸**Page 229** line −14 — 29 Mar 2023

Reading, Mass. ↝ Cambridge, Mass.

▸**Page 241** below Figure 3 — 07 Jul 2023

Fig. 3a ↝ Fig. 3(a)
Fig. 3b ↝ Fig. 3(b)
Fig. 3c ↝ Fig. 3(c)

▸**Page 268** line 5 before the exercises — 07 Jul 2023

Fundamenta Mathematica ↝ *Fundamenta Mathematicæ*

▸**Page 276** line 1 of step A4 — 01 Dec 2022

`LINK(Q1)` ← `Q` ← `LINK(Q)`, ↝ `Q` ← `LINK(Q)`, `LINK(Q1)` ← `Q`,

▸**Page 286** second table entry for time 0180, column `D2` — 07 Jul 2023

X ↝ 0

Page 310 at the "root" of Fig. 18(a) — 13 Sep 2022

Charles ↝ Charles III

Page 371 line 1 of exercise 11 — 17 Nov 2023

delete '(R. C. Prim, . . . 1389–1401.)'

▸**Page 376** line 2 of exercise 1 — 07 Jul 2023

simple oriented path ↝ simple oriented path or cycle

Page 388 line 4 after (7) — 15 Jan 2023

$n - s_j \le s_j$ ↝ $n - s_j \le s_j = \max(s_1, s_2, \dots, s_k)$

▸**Page 406** line −6 — 07 Jul 2023

(1919), 7 ↝ (1919), 7:3–7:33

▸**Page 408** line 2 of exercise 2 — 07 Jul 2023

r-sided polygon ↝ r-sided convex polygon

▸**Page 421** line −16 — 07 Jul 2023

338–343 ↜↝ 338–346

▸**Page 433** line 14 — 07 Jul 2023

(Syracuse, N.Y.: 1962) ↜↝ (Syracuse, N.Y.: 1962), 74–75

▸**Page 457** line 7 — 29 Mar 2023

Reading, Mass. ↜↝ Cambridge, Mass.

▸**Page 458** line 7 — 07 Jul 2023

61–70 ↜↝ 61–79

▸**Page 461** line 24 — 07 Jul 2023

578–588 ↜↝ 578–589

Page 462 line 9 — 07 Jul 2023

Press, 1969) ↜↝ Press, 1969), 1–75

▸**Page 462** line 12 — 07 Jul 2023

567–643 ↜↝ 547–643

Page 465 date of the Sherlock Holmes quote — 07 Jul 2003

(1888) ↜↝ (1914)

Page 466 new answer for Notes on the Exercises — 24 Nov 2023

2. They help you to lodge ideas in your brain.

Page 469 and 470 — 07 Feb 2024

[renumber equations formerly numbered (5) through (19) to be (6) through (20)]

Page 471 append a new paragraph to answer 1 — 01 Mar 2022

Note: We might also say that $a_2+\cdots+a_0=\sum_{j=2}^{0}a_j$, and that $\sum_{j=p}^{q}a_j+\sum_{j=q+1}^{r}a_j=\sum_{j=p}^{r}a_j$, using the left half of (1). On the other hand, we should *not* replace $\cdots$ using the right half of (1)! The sum $\sum_{2\le j\le 0}a_j$ is 0, by (2).

Page 472 new copy for end of answer 30 — 01 Jul 2022

exercise 46.) ↜↝ exercise 46. Cauchy's sums were extended to integrals by V. Bouniakowsky [*Mém. Acad. St.-Pétersbourg* (7) **1** (1859), No. 9, 4] and H. A. Schwarz [*Acta Societatis Scientiarum Fennicæ* **15** (1885), 315–362].)

Page 473 last line of answer 35 — 07 Feb 2024

sup(sup ↜↝ max(sup

▸**Page 481** line −2 of answer 20 — 19 Jan 2023

less than ↜↝ at most

▸**Page 485** line 3 of answer 22 — 19 Jan 2023

$(-1)^{k-1}$ ↜↝ $(-1)^k$

Page 486 replacement for answer 30 — 29 Mar 2023

30. Apply (6) and (19) to get

$$\sum_{k\ge 0}\binom{n+k}{n-m-k}\binom{-k-1}{k}\frac{1}{k+1}.$$

Now we can apply Eq. (26) with $(r,s,t,n)=(-1,2n-m,1,n-m)$, obtaining

$$\binom{n-1}{n-m}.$$

This result is the same as our previous formula, when n is positive; but when $n=0$ the answer we have obtained is correct while $\binom{n-1}{m-1}$ is not. Our derivation has a further bonus, since the answer $\binom{n-1}{n-m}$ is valid for $n\ge 0$ and *all* integers m.

▸**Page 486** line 2 of answer 31 — 29 Mar 2023

38–57 ↝ 37–57

▸**Page 489** line 1 of answer 58 — 29 Mar 2023

Arithmetik ↝ *Arithmetik* **2**

▸**Page 490** line 8 of answer 62 — 29 Mar 2023

285–289 ↝ 284–289

▸**Page 492** replacement for answer 9 — 01 Mar 2022

9. $(n=0?\ 0:\ -1/n)$.

Page 493 new answer for end of section 1.2.7 — 01 Mar 2022

25. $H_n^{(0,v)}=\sum_{k=1}^n k^{1-v}=H_n^{(v-1)}$ and $H_n^{(u,0)}=\sum_{k=1}^n H_k^{(u)}$; so the identity generalizes (8). [See L. Euler, *Novi Comment. Acad. Sci. Pet.* **20** (1775), 140–186, §2.]

Page 494 new answer for exercise 31 — 29 Mar 2023

31. Use exercise 11.

▸**Page 495** line 1 of answer 37 — 15 Nov 2023

1 (December ↝ **1**, 4 (December

▸**Page 495** replacement for last line of answer 37 — 29 Mar 2023

by A. J. Schwenk [*Fibonacci Quarterly* **8** (1970), 225–234, 241].

▸**Page 497** line 5 of answer 11 — 29 Mar 2023

Invention Nouvelle en Algébre ↝ *Invention Nouvelle en l'Algebre*

Page 497 in answer 17 — 29 Mar 2023

[change k to $k\ge 0$ in the last two sums]

▸**Page 498** replacement for reference at end of answer 21 — 29 Mar 2023

[See K. Knopp, *Theory and Application of Infinite Series*, 2nd ed. (Glasgow: Blackie, 1951), Section 66.]

▸**Page 500** replacement for line 3 of answer 10 (because $B_1=\frac12$) — 29 Mar 2023

$$H_M-\sum_{m=1}^M\left(1-\frac{m}{M}\right)^n m^{-1}=H_n+\sum_{k=1}^n\left(\binom{n}{k}-1\right)B_kM^{-k}k^{-1}-\frac{n-1}{M}.$$

Page 502 replacement for line 5 of answer 21 — 25 Apr 2023

$p \ge \frac{1}{2}$. Still further work yields the upper bound $\exp\left(-\sum_{k\ge1} 2^{2k-1}\epsilon^{2k}/(2k^2-k)\right) \le \exp(-2\epsilon^2 n)$, valid for all p.)

Page 502 append new sentence to answer 22(b) — 22 May 2023

The slightly more complicated bound $e^{-\delta^2/(2+\delta)}$ can be used for all $\delta \ge 0$, because $2\delta/(2+\delta) \le \ln(1+\delta)$.

▸**Page 502** line 2 of answer 22(c) — 29 Mar 2023

it is $1/2$ ↜ it is $< 1/2$

▸**Page 504** line 6 of answer 7 — 29 Mar 2023

122–158 ↜ 122–138

▸**Page 520** line −3 of answer 19 — 29 Mar 2023

Philomathique ↜ *philomatique*

▸**Page 523** in answer 13 — 29 Mar 2023

$X[i] = -j$ if and only if i goes ↜ $X[i] = -j$ if and only if no number $> m$ goes to i under π, but i goes

Page 524 append a sentence to answer 21 — 10 Apr 2023

(See also answer 1.2.5–21.)

▸**Page 525** last line of answer 23 — 10 Nov 2022

373 ↜ 343

Page 527 new answer — 24 Mar 2023

38. (a) Obviously $\pi = \tau^- \pi\tau \iff \tau\pi = \pi\tau$. (b) π^τ takes $b_{ij} \mapsto a_{ij} \mapsto a_{i(j+1)} \mapsto b_{i(j+1)}$, if we regard $i(l_i{+}1)$ as $i1$. (c) One of the $6 \cdot 3 \cdot 2$ solutions is $\tau = (1\,8)(2\,7)(3\,6)(4\,5)$. Others are $\tau = (2\,3)(7\,8)$; $\tau = (1\,6\,2\,8\,3\,7)$. (d) None. (e) True. (f) True. [See E. Netto, *The Theory of Substitutions and its Applications to Algebra* (1892), §46.]

Page 536 line 3 of answer 4 — 07 Jul 2023

divide a polygon ↜ divide a convex polygon

Page 537 line −4 — 07 Jul 2023

$\sum_{n,m}$ ↜ $\sum_{m,n\ge0}$

Page 538 lines 3 and 4 of answer 5 — 07 Jul 2023

is impossible, since . . . yet ↜ is impossible: It means that p_i must go off before p_j is put on, and then p_k, yet

Page 562 bottom line — 13 Sep 2022

Charles's ↜ Charles III's

▸**Page 565** line 4 of answer 13 — 07 Jul 2023

go on to T6 ↜ go on to T6′

▸**Page 576** beginning of answer 14, 15 — 01 Nov 2022

Hussain Zedan ↜ Hussein Zedan

Page 579 new paragraph at end of answer 11 — 17 Nov 2023

References: V. Jarník, *Acta Societatis Scientiarum Naturalium Moravicæ* **6** (1930), 57–61; R. C. Prim, *Bell System Technical Journal* **36** (1957), 1389–1401.

Page 586 in lines 3 and 4 of answer 4 — 07 Jul 2023

$n \times n$ solution(s) ↝ $n \times n$ tiling(s)

Page 587 replacing the the last lines of answer 5 — 31 Jan 2023

In 1974, ... Chapters 1–2]. ↝ After many decades of improvements, an optimum solution has finally been found: E. Jeandel and M. Rao [*Advances in Combinatorics* (2021) 1:1–1:39] constructed a set of eleven tetrad types, using only four colors, that tiles the plane only nontoroidally. They also proved that no solution with fewer types or fewer colors is possible.

▸**Page 598** line −6 of answer 3 — 10 Nov 2022

Discrete Math. **64** ↝ *Discrete Math.* **62**

▸**Page 600** line −9 — 07 Jul 2023

(1974) ↝ (1973)

▸**Page 605** in answer 6 — 18 Nov 2023

B3 ↝ B3′, B6 ↝ B6′, B2 ↝ B2′

▸**Page 606** in answer 12 — 07 Jul 2023

B2 ↝ B2″, B3 ↝ B3″, B4 ↝ B4″, B5 ↝ B5″

▸**Page 616** last line of answer 40 — 07 Jul 2023

96–97 ↝ 96–99

▸**Page 629** page numbers for Algorithms 2.3.5H and 2.3.5Z — 07 Jul 2023

605 ↝ 604

Page 630 and following 01 Feb 2022

Miscellaneous changes to the existing index of Volume 1 are collected here, including corrections and amendments to the old entries as well as new entries that are occasioned by the new material. Thus, the lines of the full index that have changed serve also as an index to the present document. However, when a correction or amendment has caused an old index entry to be deleted, the deletion is usually not indicated.

Bouniakowsky, Viktor Iakowlewitch (Буняковскій, Викторъ Яковлевичъ), 472.
Charles III, King of England, 310, 562.
Commutative law, 165, 185.
Conjugate of a permutation, 185, 526.
Demuth, Howard B. (= Bud), 120.
Furch, Robert Otto, 121.
Gardner, Martin, 19, 80.
Gersonides, *see* Levi ben Gershom.
Grünbaum, Branko, 384.
Hald, Anders Hjorth, 103.
Jarník, Vojtěch, 579.
Jeandel, Emmanuel Christian Francis, 587.
Levi ben Gershom (רלב"ג = לוי בן גרשון), 17.
Maurolico (= Mauro Lyco, born Marulì), Francesco, 17.
Netto, Otto Erwin Johannes Eugen, 527.
Onodera, Rikio (小野寺力男), 582.
Prim, Robert Clay, 579.
Rao, Michaël, 587.
Real part of complex number, 21, 94.
Schwarz, Karl Hermann Amandus, inequality, 36, 472.
Shephard, Geoffrey Colin, 384.
Slesvig-Holsteen-Sønderborg-Glücksborg, Christian af, *see* Christian IX.
Strongly zero, 57.
Zedan, Hussein Saleh Mahmoud (حسين صالح محمود زيدان), 576.
Zorn, Max August, lemma, 547.
Zuse, Konrad Ernst Otto, 457.