F6. *If C and C' are closed cones, and if $C^* + C'^*$ is closed, we have $(C \cap C')^* = C^* + C'^*$.*

Proof. If A and B are arbitrary sets we have $A^* + B^* \subseteq (A \cap B)^*$, for if $x \in A^* + B^*$ and $y \in A \cap B$ then $x \cdot y = a \cdot y + b \cdot y \geq 0$. If A and B are arbitrary sets including 0 then $(A + B)^* \subseteq A^* \cap B^*$ by F2, because $A + B \supseteq A$ and $A + B \supseteq B$. Thus for arbitrary A and B we have $(A^* + B^*)^* \subseteq A^{**} \cap B^{**}$, hence

$$(A^* + B^*)^{**} \supseteq (A^{**} \cap B^{**})^* .$$

Now let A and B be closed cones for which $A^* + B^*$ is closed; then $A^* + B^* \supseteq (A \cap B)^*$ by F5. □

F7. *If C and C' are closed cones, and if $C + C'$ is closed, we have $(C + C')^* = C^* \cap C'^*$.*

Proof. F6 says $(C^* \cap C'^*)^* = C^{**} + C'^{**}$; apply F5 and $*$ again. □

F8. *Let S be any set of indices and let $\overline{S}$ be all the indices not in S. Also let $A_S = \{\, a \mid a_s = 0$ for all $s \in S \,\}$. Then*

$$A_S^* = A_{\overline{S}} .$$

Proof. If $b_s = 0$ for all $s \notin S$ and $a_s = 0$ for all $s \in S$, obviously $a \cdot b = 0$; so $A_{\overline{S}} \subseteq A_S^*$. If $b_s \neq 0$ for some $s \notin S$ and $a_t = 0$ for all $t \neq s$ and $a_s = -b_s$, then $a \in A_S$ and $a \cdot b < 0$; so $b \notin A_S^*$, hence $A_{\overline{S}} \supseteq A_S^*$. □

9. Definite Proof of a Semidefinite Fact

Now we are almost ready to prove the result needed in the proof of Lemma 7.

Let D be the set of real symmetric positive semidefinite matrices (called "spuds" henceforth for brevity), considered as vectors in N-dimensional space, where $N = (n+1)n/2$. We use the inner product $A \cdot B = \operatorname{tr} A^T B$; this is justified if we divide off-diagonal elements by $\sqrt{2}$. For example, if $n = 3$ the correspondence between 6-dimensional vectors and 3×3 symmetric matrices is

$$(a, b, c, d, e, f) \;\leftrightarrow\; \begin{pmatrix} a & d/\sqrt{2} & e/\sqrt{2} \\ d/\sqrt{2} & b & f/\sqrt{2} \\ e/\sqrt{2} & f/\sqrt{2} & c \end{pmatrix}$$

preserving sum, scalar product, and dot product. A real symmetric matrix $\notin D$ has a negative eigenvalue λ, and we can write

$$A = Q \operatorname{diag}(\lambda, \lambda_2, \ldots, \lambda_n)\, Q^T \tag{9.1}$$

for some orthogonal matrix Q. Thus $x^T A x = \lambda$ for the unit vector $x = Q(1, 0, \ldots, 0)^T$; it follows easily that D is a closed cone.

The cone of spuds turns out to be self-dual:

F9. $D^* = D$.

Proof. If A and B are spuds then $A = X^TX$ and $B = Y^TY$ and $A \cdot B = \operatorname{tr} X^TXY^TY = \operatorname{tr} XY^TYX^T = (YX^T) \cdot (YX^T) \geq 0$; hence $D \subseteq D^*$. (In fact, this argument shows that $A \cdot B = 0$ if and only if $AB = 0$, for any spuds A and B, since $A = A^T$.)

Conversely, assuming (9.1), let $B = Q \operatorname{diag}(1, 0, \ldots, 0)\, Q^T$; then B is a spud, and

$$A \cdot B = \operatorname{tr} A^TB = \operatorname{tr} Q \operatorname{diag}(\lambda, 0, \ldots, 0)\, Q^T = \lambda < 0\,.$$

So A is not in D^*; this proves $D \supseteq D^*$. □

Let E be the set of all real symmetric matrices such that $E_{uv} = 0$ when $u — v$ in a graph G; let F be the set of all real symmetric matrices such that $F_{uv} = 0$ when $u = v$ or $u \not— v$. The Fact stated in Section 7 is now equivalent in our new notation to

Fact. $(D \cap E)^* \subseteq D + F$.

Proof. We have $D + F = D^* + E^*$ by F8 and F9. The argument in the Appendix below proves that $D + F$ is closed. Hence $(D \cap E)^* = D + F$ by F6. □

10. Another Characterization

Remember ϑ, ϑ_1, ϑ_2, and ϑ_3? We are now going to introduce yet another function

$$\vartheta_4(G, w) = \max\Big\{ \sum_v c(b_v) w_v \;\Big|\; b \text{ is an orthogonal labeling of } \overline{G} \Big\}\,. \qquad (10.1)$$

Lemma. $\vartheta_3(G, w) \leq \vartheta_4(G, w)$.

Proof. Suppose b is a normalized orthogonal labeling of $\overline{G}$ that achieves the maximum ϑ_3; and suppose the vectors of this labeling have dimension d. Let

$$x_k = \sum_v b_{kv}\sqrt{w_v}\,, \qquad \text{for } 1 \leq k \leq d\,; \qquad (10.2)$$

[18] M. W. Padberg, "On the facial structure of set packing polyhedra," *Mathematical Programming* **5** (1973), 199–215.

[19] Claude E. Shannon, "The zero error capacity of a channel," *IRE Transactions on Information Theory* **2**, 3 (September 1956), 8–19.

Addendum

The original paper tacitly assumed that the sum $A + B$ of closed cones is closed; but that assumption is false. For example, if C is the "three-dimensional ice cream cone" defined by $\sqrt{c_1^2 + c_2^2} \le c_3$, and if $C' = \{(b_1, b_2, b_3)^T \mid b_1 \ge b_3\}$, we have $C = C^*$ and $C'^* = \{(t, 0, -t)^T \mid t \ge 0\}$ and $(0, 1, 0)^t \in (C \cap C')^* \setminus (C^* + C'^*)$. That violates the conclusion of F6.

This counterexample was pointed out by Evan DeCorte in 2014, who also found a valid proof of the crucial Fact in Section 9: Suppose we have a convergent sequence $A_n + B_n \to C$, where $A_n \in D$ and $B_n \in F$; we shall prove that $C \in D + F$, hence $D + F$ is closed.

Let $\lambda_{n1} \ge \cdots \ge \lambda_{nN}$ be the eigenvalues of B_n. We have $\lambda_{n1} = \Lambda(B_n) \ge 0$, because $\lambda_{n1} + \cdots + \lambda_{nN} = \operatorname{tr} B_n = 0$. We also have $\Lambda(B_n) \le \Lambda(A_n + B_n)$ by (6.2); furthermore $\Lambda(A_n + B_n) \le M$ for some M and all n, because of the convergence. Hence $\lambda_{nN} \ge -MN$, and all eigenvalues of B_n are uniformly bounded. Therefore there is a subsequence in which all eigenvalues λ_{ni} converge to limiting values λ_i. It follows that $B_n \to B \in F$ and $A_n \to C - B \in D$.

Further information can be found in A. Galtman, "Spectral characterizations of the Lovász number and the Delsarte number of a graph," *Journal of Algebraic Combinatorics* **12** (2000), 131–143.

Problem P6 has been solved for $p = 1/2$ by Uriel Feige and Robert Krauthgamer, "The probable value of the Lovász–Schrijver relaxations for maximum independent set," *SIAM Journal on Computing* **32** (2003), 345–370.