[see lecture notes in folder at back of notebooks]

1963-64 Ma 108 a.

1	T	Intro., Axioms 1-9
2	W	Rat. numbers; $ae^2+be+c \neq 0$ for integers a, b, c
3	Θ	Supremum; cons. of axiom 10
4	F	Complex numbers; the exponential series.
8	T	Homework 1. 1-4 1-6 1-7 reals 1-9 1-14 1-16 equal 1-17 1-19 1-23
9	W	Fin. Homework; cont. complex nos.
10	Θ	Complex logarithms and powers. When is $(e^z)^w = e^{zw}$?
11	F	Complex trig. Begin chapter 2.
15	T	Homework 2. 1-24 1-25 1-27 1-28 1-33 1-36 1-37 1-38 1-39 1-40
16	W	Finish homework. The set $R = \{x \mid x \notin x\}$. Relations; composition
17	Θ	Functions as special relations. Identities with image and inv. image.
18	F	Equipollence. The Cantor-Bernstein theorem.
22	T	Homework 3. 2-1 2-2 2-4 2-5 Checkerboard graph
23	W	Finish homework. Graphs. Finish chapter 2
24	Θ	Open, closed sets on real.ˢ Bolzano-Weierstrass' theorem.
25	F	Closures. Quiz: 1 Countable ∴ ($\{ \subseteq$ pos. int$\}$, $\{$ext. prods $\subseteq$ integers$\}$). 2. All [illegible]
29	T	Homework 4. 2-7 2-9 2-10 2-13 R arb. S integs. 2-14 2-15 find nec + suff that $(A \times B) \times C = A \times (B \times C)$ 3-2 3-3
30	W	Finish homework and quiz. Start metric spaces and E_n.
31	Θ	Discuss topology in metric spaces.
1	F	Compact ⇒ all unbdd seq. acc. pt. ⇔ count. compact. Quiz: 1. f, $f^{-1}(U, \cap^1)$. 2. F6 = H F ⇒ 6 = H
5	T	Homework 5. 3-10 3-12 3-14 3-18 3-19 3-20 3-21. Also discuss the quiz.
6	W	Countable cpt ⇒ comp. Quiz about boundary.
7	Θ	Continuity and compactness: definitions, f(compact) = compact.
8	F	Continuity and compactness; and homeomorphisms. Give out take-home
12	T	} no class. Test: 1. f, f^{-1}(certain sets), $f(z) = \sin z$. 2. algebraic #'s countable
13	W	} 3. Subgroups of reals open closed. 4. sup AB = sup A sup B 5. Cantor set perfect. 6. Knaster-Kuratowski comp. [illegible]
14	Θ	Connectedness.
15	F	Limits.
19	T	Homework 6. 4-1/2 gen (1) 4-4 4-6 4-8 4-10
20	W	Unif continuity
21	Θ	Monotonicity and ratio tests; comparison series
22	F	Simple tests for conv. Quiz: 1. lim exp(-1/x²) cos² x [illegible] 2. [illegible]

	H1	H2	H3	Q1	H4	Q2	H5	Q3	Midterm	exam.
Aschbacher, M.G.	2.6	0.1	0.1	2.0	1.1	0.3	4.2	4.8	1.4	A
Bigelow, R.H.	5.5	5.6	0.1	0.6	4.0	1.7	3.3	2.7	0.9	A
Bornzin, G.L.	0.6	3.4	1.4	3.0	2.6	0.2	1.1	0.9	-1.3	C
Cohen, M.	2.6		2.4			2.8		4.5	1.7	A
Frenk, R.S.	1.5	3.3	1.5	2.6	3.0	0.5	2.7	0.4	-1.5	C
Gorostiza, L.	3.2	5.2	0.2	2.9	2.8	3.0	3.5	2.6	-4.3	D
Kabell, J.A.	1.5	3.9		0.6	1.3	3.6	0.2	5.4	0.1	B
Miller, L.R.	1.5	3.9	2.3	4.2	3.7	0.4		4.7	0.3	B
Morse, S.		4.1	0.0	1.8	0.7	1.2	0.7	0.8	-1.3	C
Roberts, P.H.	0.3	3.2	1.0	2.0	3.7	0.0	-1.7	3.4	0.4	B
Shultz, F.W.	3.3	0.5	0.2	5.5	2.6	1.1		4.7	2.4	A
Sirelson, V.L.	2.5	3.8	2.1	2.0	3.1	0.0	-3.0	0.1	-2.3 +	C
Stanley, R.P.	5.6	5.4	3.1	3.8	3.8	0.7	5.0	2.6	3.8	A
Vogel, R.A.	0.7	2.5	0.2	0.1	2.8	0.2	0.6	1.3	1.1	A
Williams, R.J.	0.2	5.9	0.6	1.4	3.0	0.7		1.0	1.5	A
Wolf, M.L.	2.7	1.9	0.4	5.7	3.0	0.9	-1.2	0.0	0.9	A
Wright, N.J.	5.9	3.4	2.2	2.7	4.8	1.3		3.5	-3.9	D
Median	0	0.1	0	0.6	0	0.2	0.2	1.0	0.3	

More detailed breakdown.

		H1									Q1			Q2												Midterm					
		1	2	3	4	5	6	7	8	9	H	pm		f'	f	FGH	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6
M	Aschbacher	0	1	1/2	1	0	0	1	1	1/2	0	0	2/5	1/6-	0	1-2ε	1/2	1/2	1	1-2ε	0	0	0	0	0	3/8	1-	4/5+	1-	1/3+	3/8
R	Bigelow	1-	1	1	1	1-	1	1	1	1	0	0	1	1	1-	1/4	1	1	1	1-	1	1	1/2	0	0	1-	1	1/3	1	1/4	1/2+
G	Bongen	1	1	1/2	0	1/2	0	1	1	1	0	0	0	1	1/5	0	1	1-	1	1/2	1-	1/2	0	0	0	1/2-	ε	1/2+	1-	5/6	ε
M	Cohen	1	1	0	1	1	1	1	0	1	—			1	1/3	1	1/2	0	1	1-2ε	0	1/2	0	0	0	1/4+	1	1-2ε	1-	2/3+	5/8
R	Frente	0	1	1/2	1-	1/2	1	1	1	1/2	0	0	1/6	2/3	3/5	0	ε	ε	1	1/2	1	1	1	0	0	1/4	1	1/3	1-	1/6	1/4
L	Arrostega	0	1	1/2	0	0	0	1	1-	1	0	0	ε	0	0	0	1-	ε	1	0	1/2	1-	1/3	0	0	0	0	0	1-	1/6	1/8
J	Kabell	0	1	1/2	1	1/2	0	1	1	1/2	0	0	1	1	3/5+	1	1	2/3	1	1-2ε	3/4	1/2	1	1/4	0	5/12	1	1/3	1-	2/3-	3/8+
L	Miller	1	1	1/2	1	1/2+	1/2-	1	1	1/2	1	1/2	2/5	1-	4/5-	0	1	1-	1	0	ε	0	0	0	0	1/2	1	1/6	1	5/6	3/8
S	Morse	—									0	0	1/2	2/3	3/10	0	1	1	1	1/2	1-	1/2	1/2	0	0	3/5	2ε	2/3	1-	ε	3/8
P	Roberts	1	1	1/2	1	0	1/2	0	1	1	0	0	2/5	1	3/5	0	1	2/3	1	1/2	1/2	0	0	0	0	1/5	1/2	1-	1	1-	1/4
F	Shultz	1	1	1/2	1	1/2	0	0	0	0	1-	1/2	1	1	1/6	0	1/3	1/4	1-	1/2	ε	1	0	0	0	3/4	1	1/2	1-	1	2/3
V	Snelson	1	1	1	1	1/2	1/3	1	1	1-	0	0	2/5	1	3/5	0	1	1-	1	1-2ε	1	1/2	ε	0	0	1/3	1	1/2	1/3	1/6	ε
R	Stanley	1	1	1	1	1	1	1	1	1	3/4	1/3	1	5/6	3/5+	ε	1	1-	1-2ε	1-2ε	1	1	3/4	1-	1	3/4	1-	1-	1	5/6	1
R	Vogel	1-	1	1/2	1-	1/2+	0	1	1	1	0	1/2	ε	1-3ε	4/5-	0	1/2	1/2	1	1/2	1	1	ε	0	0	3/5	1	1/2	1	5/6	3/8
R	Williams	1	1	1/2	1	1	0	0	1	1	1	0	2/5	1/6-	1/6	1-2ε	ε	ε	1	0	0	0	0	0	1	3/4-	1	2/3	1	2/3	3/8+
M	Wolf	1/2	1	1/2	1	0	0	0	1	1-	1-	1-	0	0	2/5	1/3	1	0	1	1/2	0	ε	ε	2ε	1/2	1/3	1	1-2ε	1	5/6-	1/4
N	Wright	0	1	1/2	0	0	0	0	0	1-	0	ε	0	0	0	1/2	1/2	1/2	1	ε	0	0	1-2ε	0	0	1/2+	ε	0	1/3-	0	1/4
	Class average	65	99	56	80	47	33	68	80	80	29	18	4[illegible]	66	40	29	70	53	98	51	51	50	30	8	15	50	74	53	89	54	38

Homework Scores (2nd term)

	1	2	3	4	5	6	7	8	9	
M Asch	0.9	1.0	2.8	6.5	10.2	2.5	-0.6	0.5	3.1	A+
R Berg	-2.7	0.8	2.3	-0.5	-0.3	3.6	0.7	1.4	2.9	A
R Frenk	-0.9	0.4	1.0	-0.5	-0.2	1.6	4.5	-0.4	-1.2	A-
J Kabell	**	**	**	**	**	-3.6	**	-1.0	**	F
R Lane	10.7	0.3	**	**	-2.5	-2.3	0.0	-0.9	-0.2	C+
L Miller	-3.4	-4.2	-6.9	-0.5	-3.2	-0.5	-4.0	-1.1	0.0	C
F Schultz	-2.9	2.1	**	**	-3.2	-0.6	0.4	1.1	**	C-
V Snelson	-2.1	-1.1	-1.8	-1.3	-3.2	-2.2	-2.7	-0.7	-2.5	C-
R Stanton	2.9	0.8	4.5	-1.3	-0.2	1.8	0.6	1.2	1.9	A+
R Vogel	-0.7	-1.4	-2.8	-0.9	**	-2.5	**	-0.4	0.5	C+
R Williams	-1.0	**	**	-0.6	2.2	-0.3	-3.9	0.4	-1.6	C+
M Wolf	-2.9	**	**	**	1.9	**	**	**	**	F
L Gorostiga	2.2	1.3	0.8	-0.8	-1.6	2.6	4.9	0.0	0.0	A

	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	HW	Q	Term
R Bigelow	0.5	0.1	0.3	0.3	4.6	0.8	1.9	0.3	A	A+	A+
R Frenk	0.3	-0.4	-0.4	-0.9	0.7	0.8	-1.2	-0.9	A	B	A-
J Kabell	0.5	-0.4	-0.4	-0.3	-2.7	-0.1	-0.8	0.2	D D+	~~C+~~ C+	C- ~~D+~~
L Miller	-0.8	0.1	-0.5	0.3	-1.8	-1.0	1.9	0.3	B+	B+	B
F Schultz	-0.2	0.1	0.4	-0.3	-0.6	0.8	-0.2	0.2	A+	A-	A+
V Sirelson	-0.6	0.1	0.3	0.3	0.1	-0.1	-0.3	0.2	B	A	B+
R Stanley	0.5	0.1	0.3	0.2	4.0	0.8	-0.4	0.2	A+	A	A+
R Vogel	-0.2	0.1	0.5	0.2	-0.4	**	**	0.0	~~B-~~ C	~~B+~~ C+	C+
R Williams	0.5	0.1	-0.4	0.0	-3.7	-0.1	-0.8	0.2	B-	B+	B
M Wolf	-0.6	0.1	0.3	0.3	**	-1.0	0.5	**	F	C	D
L Gorostiza	0.2	0.1	-0.2	-0.3	0.0	-1.0	-0.7	-0.9	B	B	B

L8384

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238-420

July 7, 1969

Dear Ed,

Thanks for the copies of yours and Don Knuth's papers on plane partitions. I too have been trying to tie your work up with Schur functions, etc. An interesting paper on this topic which you may not be aware of is: Philip Hall, The Algebra of Partitions, Proc. 4th Canad. Math. Congr. Banff. 1957, p. 147-159 (1959). Here are a couple of results and a conjecture in which you may be interested (perhaps you have discovered them yourself).

(1) Let λ be a partition of n, and let $h_1, h_2, \ldots, h_n$ be the hook lengths of the shape λ (no. of squares directly to the right or directly below a square in the shape λ). Let $c_1, \ldots, c_n$ be the signed distance of the squares in the shape λ from the main diagonal. Let $\lambda_1, \lambda_2, \ldots$ be the parts of λ. Then the generating function for row-strict plane partitions with shape λ and largest part $\leq m$ is

$$x^{\sum \binom{1+\lambda_i}{2}} \frac{(1-x^{m+c_1})(1-x^{m+c_2})\cdots(1-x^{m+c_n})}{(1-x^{h_1})(1-x^{h_2})\cdots(1-x^{h_n})}$$

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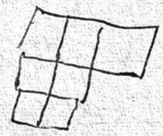
EX. λ : $3+2+2$ $\lambda_1 = 3,\ \lambda_2 = 2,\ \lambda_3 = 2$



shape

5	4	1
3	2	
2	1	

hook lengths

0	-1	-2
1	0	
2	1	

c_i's

generating fn: $$x^{12}\frac{(1-x^{m-2})(1-x^{m-1})(1-x^{m})^2(1-x^{m+1})^2(1-x^{m+2})}{(1-x^5)(1-x^4)(1-x^3)(1-x^2)^2(1-x)^2}$$

~~Also the~~ By taking the shape of λ to be an $r\times c$ rectangle, it is easy to derive from this the generating function for r-rowed c-columned plane partitions.

Also the number of "reverse plane partitions" (non-decreasing in rows and columns) with shape λ has the gen. fn. $x^n/(1-x^{h_1})(1-x^{h_2})\cdots(1-x^{h_n})$ (no restriction on part size). The number of row and column strict reverse plane partitions of shape λ has the gen. fn. $x^{h_1+h_2+\cdots+h_n}/(1-x^{h_1})\cdots(1-x^{h_n})$.

(2) The coefficient of $a^t x^n$ in the expansion of $\prod_{i=1}^{r}\prod_{j=1}^{m}(1-ax^{i+j-1})^{-1}$ is the number of r-rowed plane partitions of n, no part $> m$, such that $t=\sum_{k=1}^{r}\alpha_k$, where α_k is the number of entries in the k^{th} row $\geq k$.

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If $m = \infty$, the coef. of $a^t x^n$ is also the number of r-rowed plane partitions of n with <u>trace</u> t (trace = sum of main diagonal elements, e.g., trace $\begin{array}{ccc} 4 & 4 & 2 \\ 3 & 1 & \\ 3 & 1 & \end{array} = 4 + 1 = 5$).

Some typical corollaries: ~~The~~. If ~~$0 \le t \le m$~~ $0 \le n \le t$, then the number of plane partitions of $n + t$ with trace t is the coef. of x^n in $\prod_{i=1}^{\infty} (1-x^i)^{-(i+1)}$.

Write
$$\prod_{n=1}^{\infty} (1-ax^n)^{-n} = \sum_{n=0}^{\infty} \frac{x^n C_n(x)\, a^n}{(1-x)^2 \cdots (1-x^n)^2}.$$

Then
$$C_n(x) = \sum_{\substack{\text{partitions } \lambda \\ \text{of } n}} x^{\sum \lambda_i(\lambda_i - 1)} \left[\frac{(1-x^n)(1-x^{n-1}) \cdots (1-x)}{(1-x^{h_1}) \cdots (1-x^{h_n})} \right]^2,$$
where λ_i are the parts of λ and h_i the hook lengths.

(3) It follows from your Thm. B that
$$\sum_{\lambda} e_\lambda = \left[\prod_{i=1}^{\infty} (1-x_i)^{-1} \right] \left[\prod_{1 \le i < j} (1 - x_i x_j)^{-1} \right],$$
where the sum is over all partitions λ and e_λ are the Schur functions. My conjecture concerns inverting this expression.

<u>Conjecture</u> $\displaystyle \frac{1}{\sum_\lambda e_\lambda} = \sum_\lambda \alpha_\lambda e_\lambda$, where

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$$\alpha_\lambda = \begin{cases} 0, & \text{if } \lambda \text{ is not self-conjugate} \\ (-1)^d, & \text{if } \lambda \text{ is self-conjugate, and } d \text{ is the no. of els on or below the main diagonal in the shape of } \lambda. \end{cases}$$

A corollary to this conjecture would be: The number of row-strict plane partitions of n with self-conjugate shape and an even no. of entries on or below the main diagonal, minus ditto except an odd no. on or below main diagonal, is the coef. of $\frac{x^n}{n!}$ in the expansion of $e^{-x-\frac{x^2}{2}}$.

This conjecture, if true, probably follows from Thm. C, but a lot of cancellation must come in which I haven't yet figured out.

I would appreciate being informed of your new results on this subject. Maybe I will have a chance to see you after the summer.

Sincerely,

Richard Stanley

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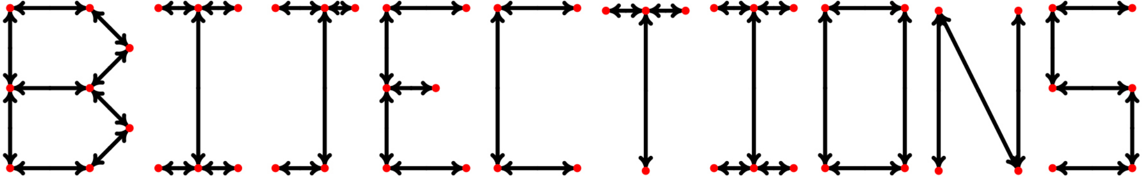

BIJECTIONS

Q114 The genesis of attribute grammars.

TeX file `gag.tex` (17167 bytes)

Unpublications

The following documents are essentially notes to myself and a few colleagues; I don't intend to polish them for publication, although they might form the nucleus of later work. I suggest that you download them only if you haven't been able to find anything more exciting to read on the World Wide Web.

Bipermutations (96.06.28)

TeX file `biperm.tex` (5785 bytes)

Squared differences of uniform deviates (95.07.18)

TeX file `sdud.tex` (1588 bytes)

An interesting bijection (95.05.17)

TeX file `ib.tex` (5047 bytes)

Solution to AMM problem 10430 (95.02.24)

TeX file `10430.tex` (2145 bytes)

Solution to AMM problem 10424 (95.02.24)

TeX file `10424.tex` (744 bytes)

Coups de grâceful graphs (20.11.16)

[PDF (109721 bytes)]

Amazing grace (21.01.01)

[PDF (149072 bytes)]

Von Neumann's 1945 manuscript on sorting (21.04.10)

[PDF (158454392 bytes)]

Baxter matrices (21.09.05)

[PDF (107937 bytes)]

Xqueens and Xqueenons (21.10.06)

[PDF (124067 bytes)]

Ambidextrous numbers (22.09.07)

[PDF (241670 bytes)]

The CVM algorithm for estimating distinct elements in streams (23.05.25)

[PDF (232859 bytes)]

Parades and poly-Bernoulli bijections (24.03.14)

[PDF (329232 bytes)]

Parades and Poly-Bernoulli Bijections

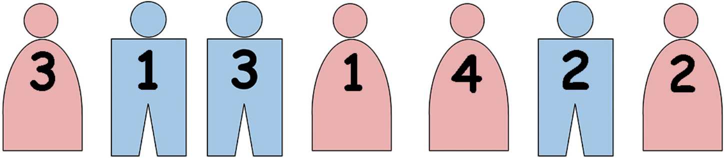
Don Knuth, Stanford Computer Science Department
Pi Day, 2024; revised 10 May 2024

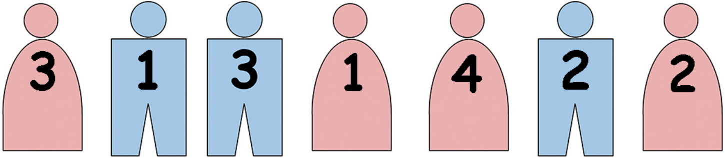
This is a story about a beautiful array of numbers that arises in an astonishing number of interesting combinatorial contexts. It's a counterexample to the hypothesis that all of the important "special numbers" were discovered long, long ago — because the earliest prominent appearance of this particular array was in 1997. It has, however, been rediscovered several times since then.

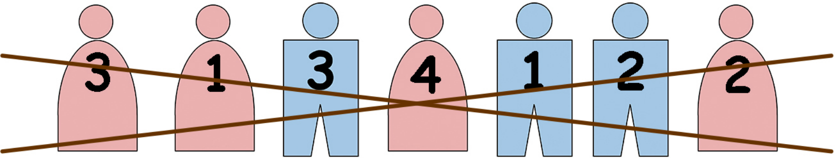
The numbers in question, which we shall call $B_{m,n}$ in this note, begin as follows:

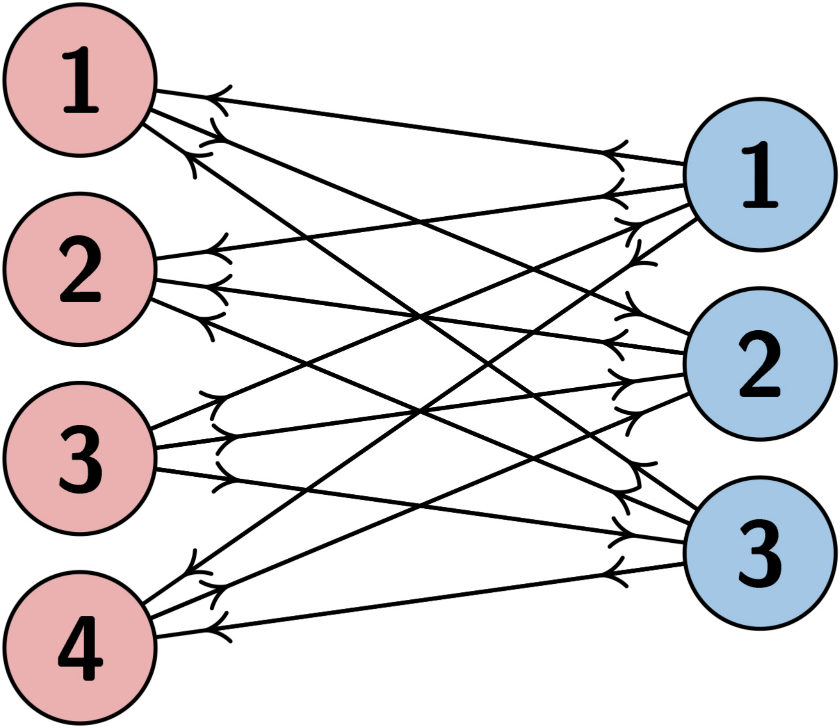
$B_{m,n}$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$m=0$	1	1	1	1	1	1
$m=1$	1	2	4	8	16	32
$m=2$	1	4	14	46	146	454
$m=3$	1	8	46	230	1066	4718
$m=4$	1	16	146	1066	6902	41506
$m=5$	1	32	454	4718	41506	329462

Notice that we have diagonal symmetry, $B_{m,n} = B_{n,m}$, throughout this table.

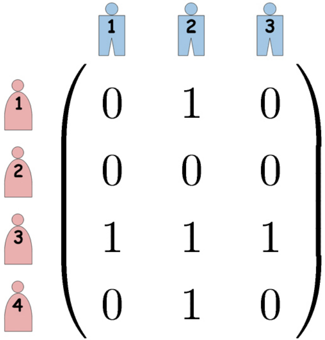

3
1
3
1
4
2
2


3
1
3
1
4
2
2


3
1
3
4
1
2
2



Acyclic orientation of $K_{4,3}$



$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

A “lonesum” matrix: This is the only matrix of 0s and 1s whose row sums are $(1, 0, 3, 1)$ and column sums are $(1, 3, 1)$.

Permutation $p_1p_2p_3p_4p_5p_6p_7$ of $\{1,2,3,4,5,6,7\}$ with

$$k-4 \;\le p_k \le k+3 \qquad \text{for } k = 1,2,3,4,5,6,7:$$

$$1\;3\;4\;5\;2\;6\;7$$

$$\begin{pmatrix} ♜ & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & ♜ & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & ♜ & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & ♜ & 1 & 1 \\ 1 & ♜ & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & ♜ & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & ♜ \end{pmatrix}$$

(Place nonattacking rooks, avoiding the 0s.)

Catalan Numbers

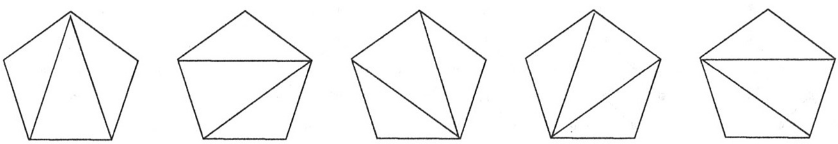
RICHARD P. STANLEY

2

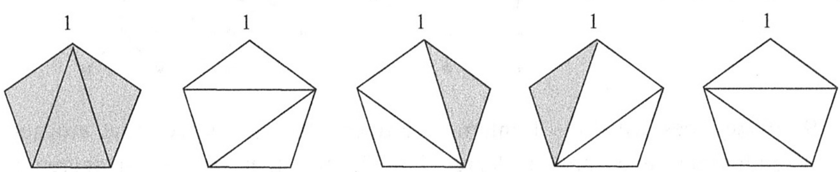
Bijective Exercises

We now come to the core of this monograph, the 214 combinatorial interpretations of Catalan numbers. We illustrate each item with the case $n = 3$, hoping that these illustrations will make any undefined terminology clear. Some of this terminology also appears in the Glossary. Solutions, hints, and references are given in the next section. In some instances two different items will agree as sets, but the descriptions of the sets will be different. Readers seeking to become experts on Catalan numbers are invited to take each pair (i_1, i_2) of distinct items and find a bijection (valid for all n) from the sets counted by i_1 to the sets counted by i_2, so $214 \cdot 213 = 45582$ bijections in all!

1. Triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors.



2. Total number of triangles with vertices $1, i, i+1$, $2 \leq i \leq n+1$, among all triangulations of a convex $(n+2)$-gon with vertices $1, 2, \ldots, n+2$ in clockwise order.



2

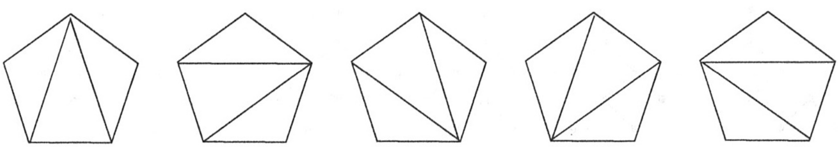
Bijective Exercises

We now come to the core of this monograph, the 214 combinatorial interpretations of Catalan numbers. We illustrate each item with the case $n = 3$, hoping that these illustrations will make any undefined terminology clear. Some of this terminology also appears in the Glossary. Solutions, hints, and references are given in the next section. In some instances two different items will agree as sets, but the descriptions of the sets will be different. Readers seeking to become experts on Catalan numbers are invited to take each pair (i_1, i_2) of distinct items and find a bijection (valid for all n) from the sets counted by i_1 to the sets counted by i_2, so $214 \cdot 213 = 45582$ bijections in all!

0. Integers between 0 and $C_n - 1$, inclusive.

0 1 2 3 4

1. Triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors.



Three Catalan Bijections

D. Knuth

REPORT No. 04, 2004/2005, spring

ISSN 1103-467X
ISRN IML-R- -04-04/05- -SE+spring

INSTITUT MITTAG—LEFFLER

KUNGL · VETENSK · AKAD · SIGILL

P127 Stable husbands.

TeX file sh.tex (15411 bytes)

P123 Efficient representation of perm groups.

TeX file erpg.tex (13113 bytes)

Unrefereed Papers

The following invited papers did not benefit from the formal refereeing mechanisms:

Q241 Randomness in music

Compressed PostScript file randomness.ps.gz (254K bytes)

Q204 Three Catalan bijections

Compressed PostScript file tcb.ps.gz (70K bytes); PDF file tcb+.pdf (259K bytes)

Q201 Robert W Floyd, In Memoriam

Compressed PostScript file floyd.ps.gz (741K bytes)

Q171 Teach Calculus with Big O (unabridged version)

TeX file ocalc.tex (7860 bytes)

Q145 Foreword to A=B by Petkovsek, Wilf, and Zeilberger.

TeX file pwz.tex (1487 bytes)

Three Catalan Bijections

by Donald E. Knuth

Institut Mittag-Leffler and Stanford University

25 February 2005

This note contains three short programs that implement one-to-one correspondences between four kinds of combinatorial structures:

1) Ordered forests with n nodes and pruning order m;
2) Binary trees with n nodes and Strahler number m;
3) Nested strings (Dyck words) of length $2n$ and log-height m;
4) Kepler towers with n bricks and m walls.

In each case the number of structures of size n is the Catalan number $C_n = \binom{2n}{n}/(n+1)$, and — surprisingly — the bijections also preserve the parameter m.

Given a number $n > 1$, each program generates all C_n objects of one type, bijects them into objects of another type, verifies that the parameter m has not changed, and applies the inverse bijection to prove constructively that the correspondence is indeed one-to-one (at least for this value of n).

Program 1, called ZEILBERGER, converts between (1) and (2). Program 2, FRANÇON, converts between (2) and (3). And Program 3, VIENNOT, converts between (3) and (4).

Incidentally, Kepler towers appear to be a completely new kind of object, recently invented by Xavier Viennot and introduced here for the first time. Simple bijections between (2) and (4), or between (1) and (4), are not yet known, although complex bijections could of course be obtained by composing those given here.

A Catalan staircase alternative tableau (in Maule 79th SLC)

email: X.V.
first name (at) name (dot) org

You are on the first page of the of X.V. (new) website: www.viennot.org

During the reconstruction the "old" website www.xavierviennot.org is still active.

This website contains the slides and links to the videos of the course I have given during four consecutive years at IMSc, the Institute of Mathematical Science, Chennai, India.

"The Art of Bijective Combinatorics"

The combinations of the slides, the videos and this website form a kind of "video-book" which is in construction, even if all the slides and links to the 77 lectures are there.
Preface

Part I: An introduction to enumerative, algebraic and bijective combinatorics (January-March 2016)
Part II: Commutations and heaps of pieces with interactions in physics, mathematics and computer science (January-March 2017)

Part III: The cellular ansatz: bijective combinatorics and quadratic algebra (January-March 2018)
Robinson-Schensted-Knuth, Asymmetric Exclusion Process, Tilings, Alternating Sign Matrices ... under the same roof
Part IV: A combinatorial theory of orthogonal polynomials and continued fractions (January-March 2019)

photo by Sergey Dovgal, CIRM, Luminy, June 2021

Euler, Strahler and Knuth

Journées Knuth07

Agora, Université Bordeaux 1

29-31 Octobre 2007

xavier viennot

LaBRI, CNRS,

Université Bordeaux 1

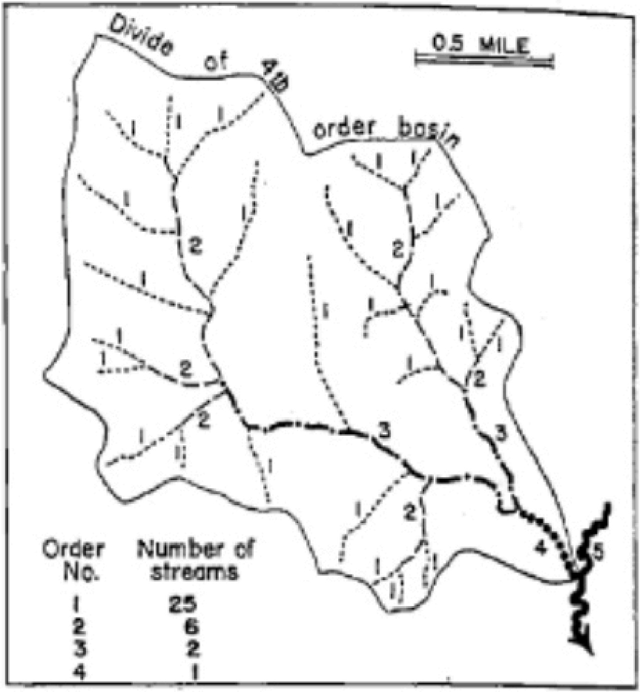
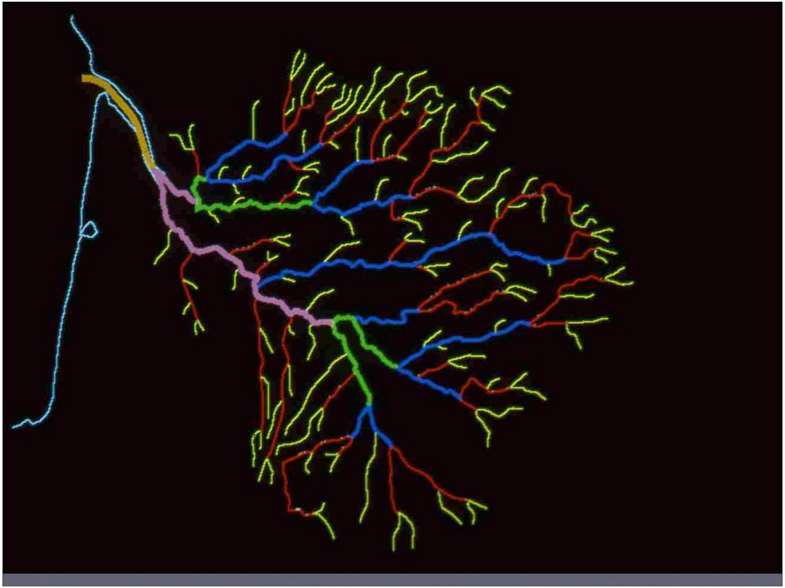
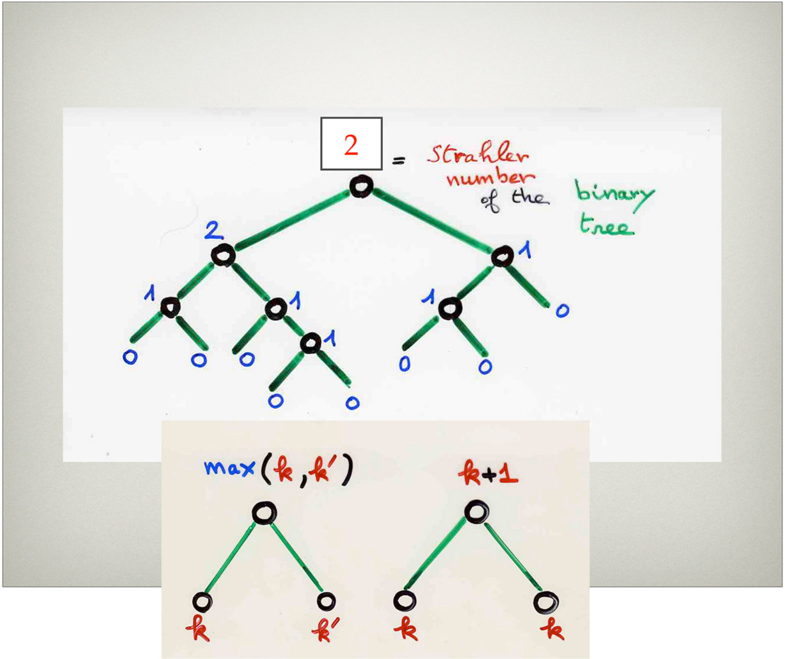


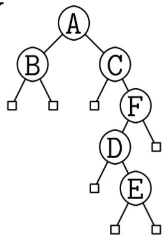
FIG. 2 – Method of designating stream orders (Strahler, 1954a, p. 344)





2
= Strahler number of the binary tree
2
1
1
1
1
1
0
0
0
0
0
0
0
0
max(k, k′)
k
k′
k+1
k
k

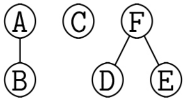
Binary tree: Either (i) empty, or (ii) a root node plus a left binary subtree plus a right binary subtree.



[There are C_n binary trees with n nodes.]

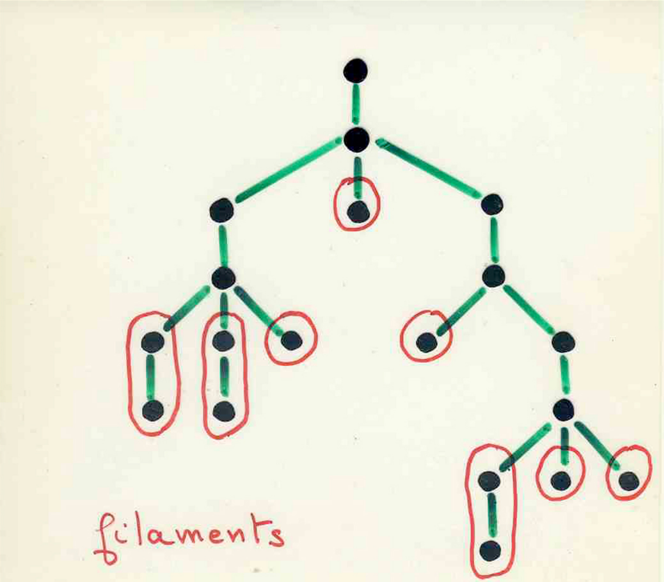
Tree (= plane tree): A root node plus a sequence of zero or more subtrees.

Forest: A sequence of zero or more trees.



[There are C_n forests with n nodes.]

Forest: Either (i) empty, or (ii) a root node plus a forest of its children plus another forest.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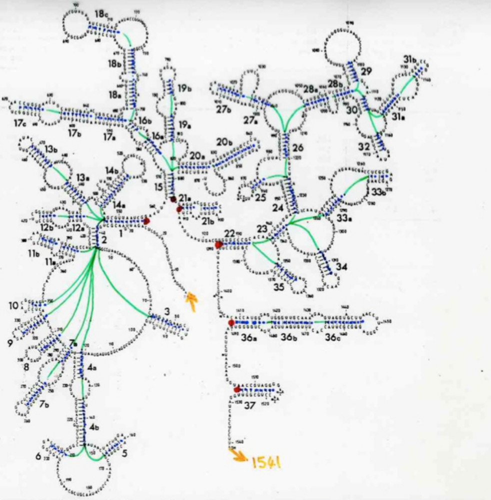

Fig. 1. *Secondary structure model of the 16-S RNA from* E. coli. This model has been fully described elsewhere [18]. The various secondary structure motifs are numbered for reference. Base-pairings 2 and 23 are included in this up-dated scheme and slight modifications have been introduced into helices 18b and 33b

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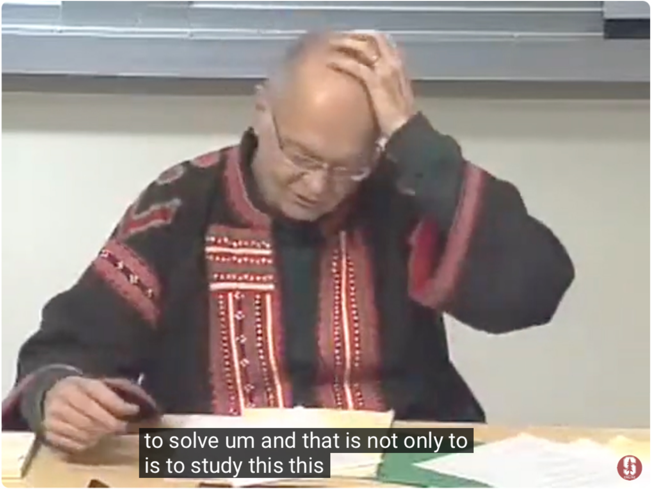

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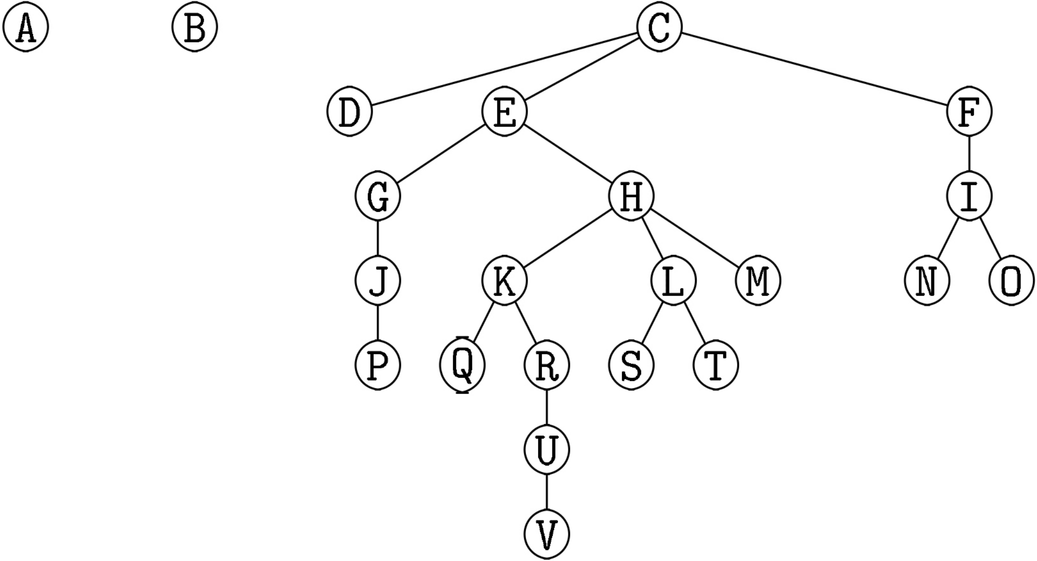
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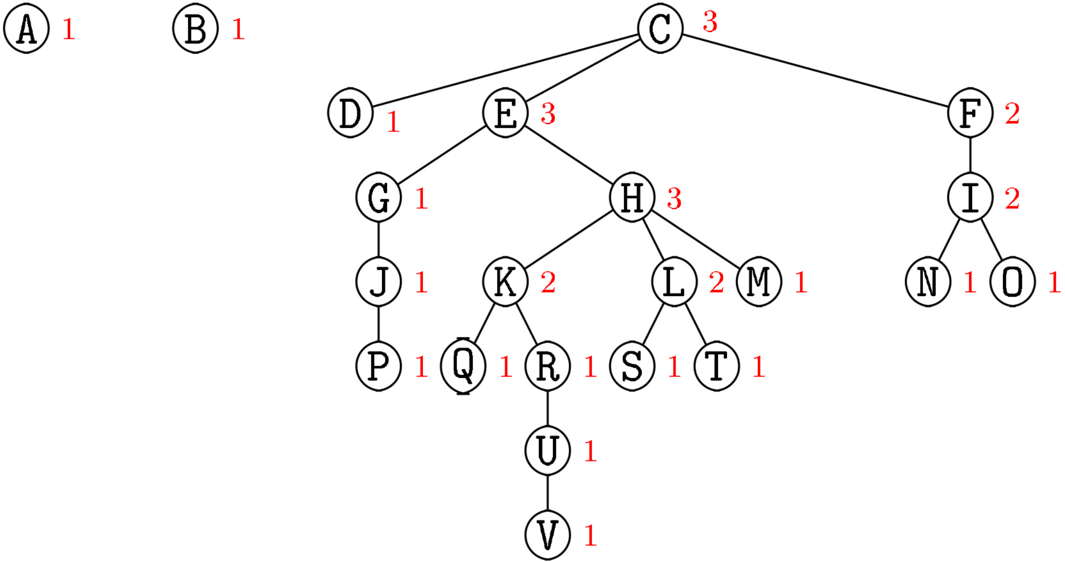

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V 1

Zeilberger's bijection proves constructively the remarkable fact, discovered by Mireille Vauchaussade de Chaumont in her thesis (Bordeaux, 1985), that the pruning order and Strahler number have precisely the same distribution, when forests and binary trees are chosen uniformly at random.

Furthermore, as we shall see, his bijection has another significant property: When forests are represented in the natural way within a computer, as binary trees with left links to the leftmost child of a node and with right links to a node's right sibling, Zeilberger's transformation preserves all of the left links: Node x is the leftmost child of y in the original forest if and only if x is the left child of y in the binary tree that is produced by Zeilberger's procedure. In particular, the number of leaves in the forest equals the number of "left leaves" in the corresponding binary tree.

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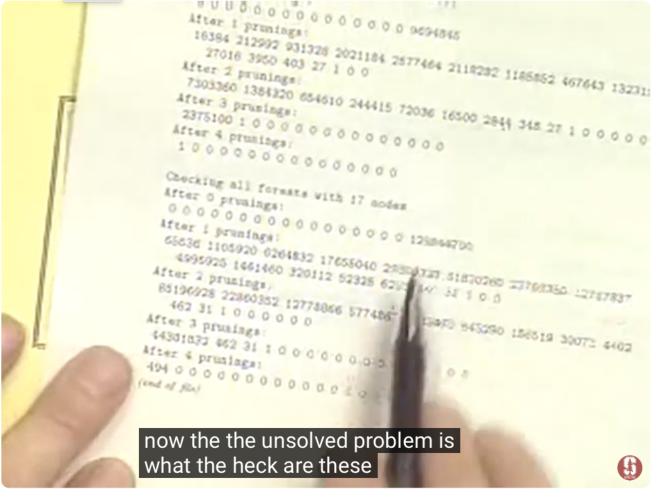

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Discrete Mathematics 82 (1990) 89–92
North-Holland

COMMUNICATION

A BIJECTION FROM ORDERED TREES TO BINARY TREES THAT SENDS THE PRUNING ORDER TO THE STRAHLER NUMBER*

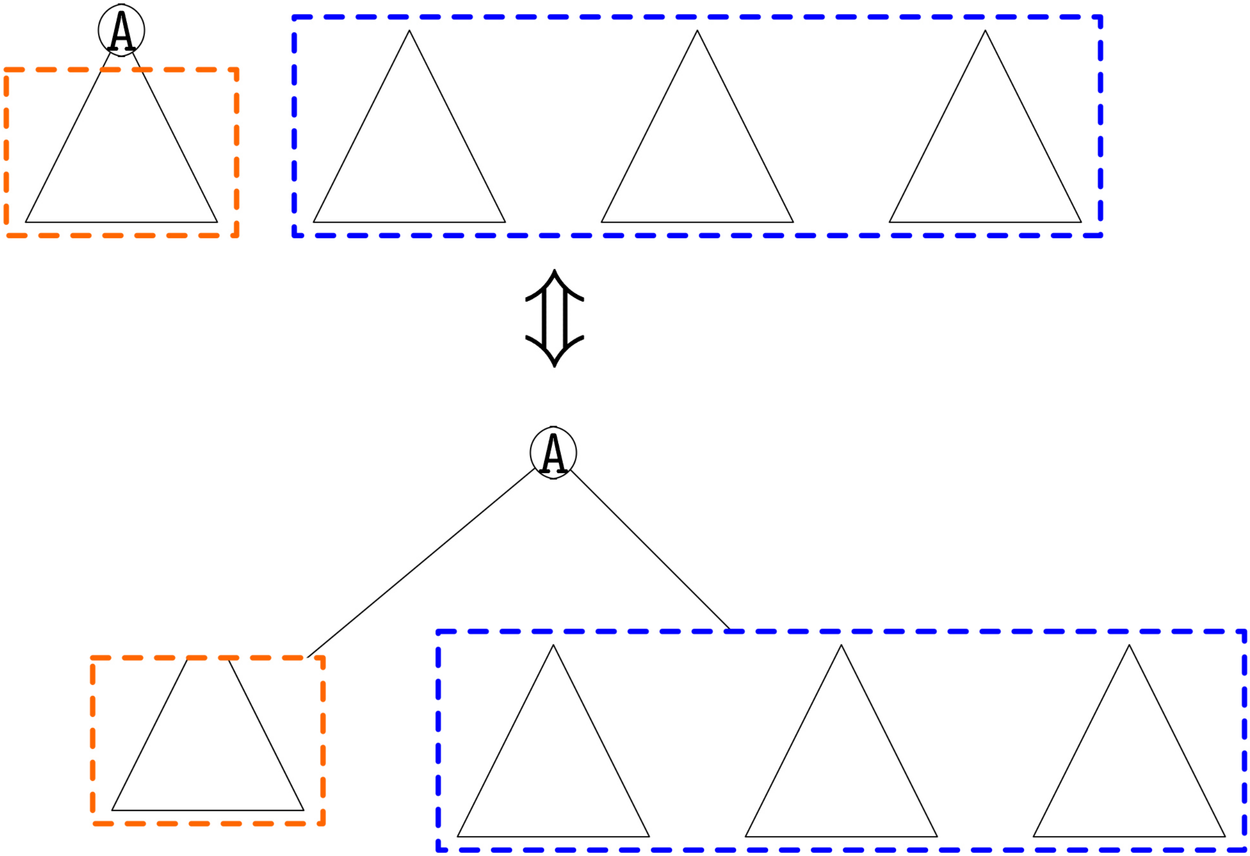
Doron ZEILBERGER
Department of Mathematics and Computer Science, Drexel University, Philadelphia, PA 19104, USA.

Communicated by I. Gessel
Received 12 December 1989

In accordance with the principle from other branches of mathematics that it is better to exhibit an explicit isomorphism between two objects than merely to prove that they are isomorphic, we adopt the general principle that it is better to exhibit one-to-one correspondence (bijection) between two sets than merely to prove that they have the same number of elements. (Richard Stanley [2], p. 11.)

It is well known (e.g. [3], p. 60) that the number of ordered trees with n vertices equals the number of complete binary trees with n leaves. Vauchaussade de Chaumont and Viennot [4, 6] (see also [3], ch. 3, ex. 6 (p. 103)) discovered an interesting refinement of this fact. They proved that for any integers n and k, the number of ordered trees with n vertices and pruning order k equals the number of complete binary trees with n leaves and Strahler number k. In this communication I construct a bijection whose "shadow" is this result, thus giving a "bijective proof" of the Vauchaussade–Viennot result and thereby solving their ten-bottles-of-wine problem [5]. This problem was also solved, independently, by Bender and Canfield [1].

First, definitions! It will be convenient to adhere to Schutzenberger's philosophy of viewing combinatorial objects such as trees as words in an appropriate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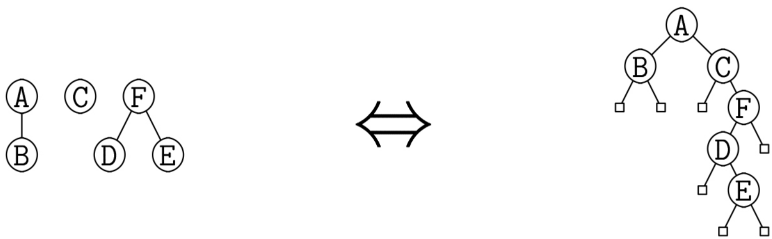
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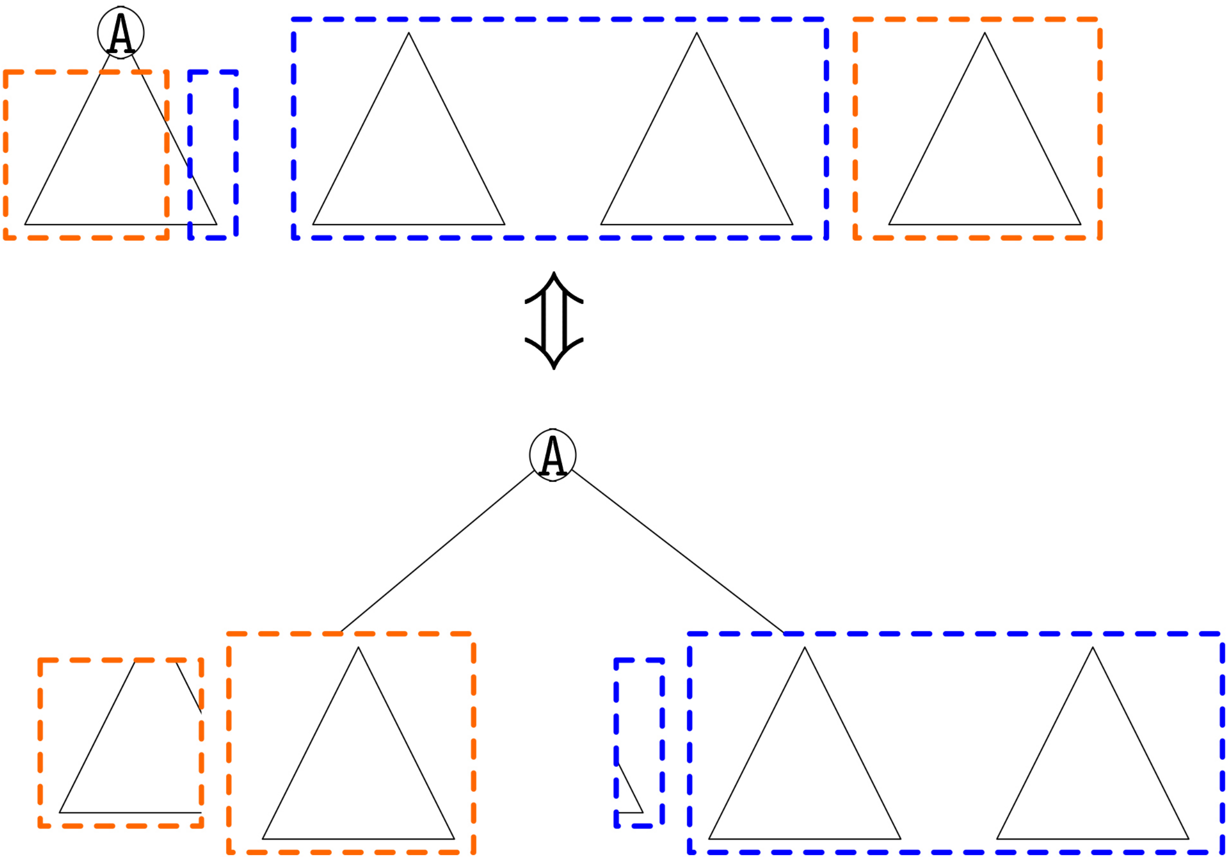
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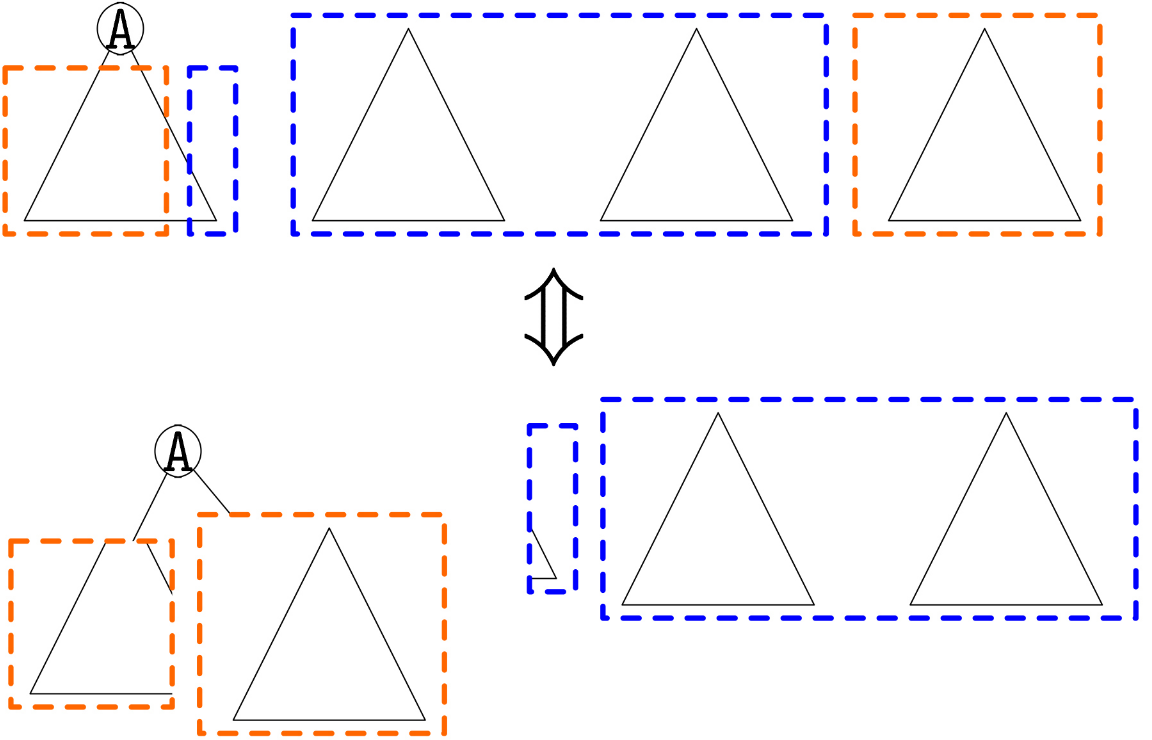
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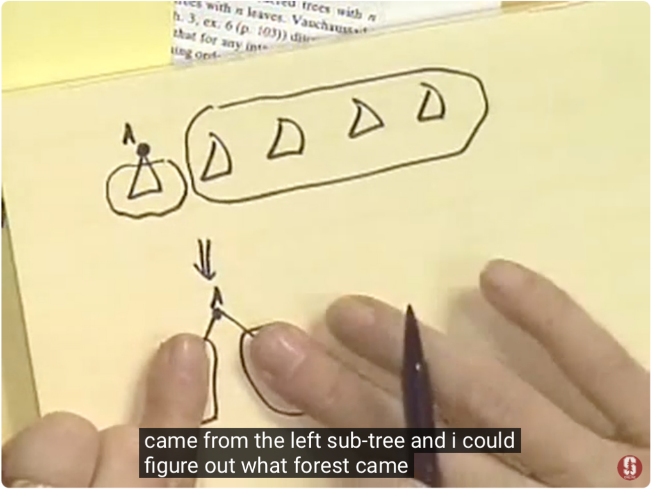

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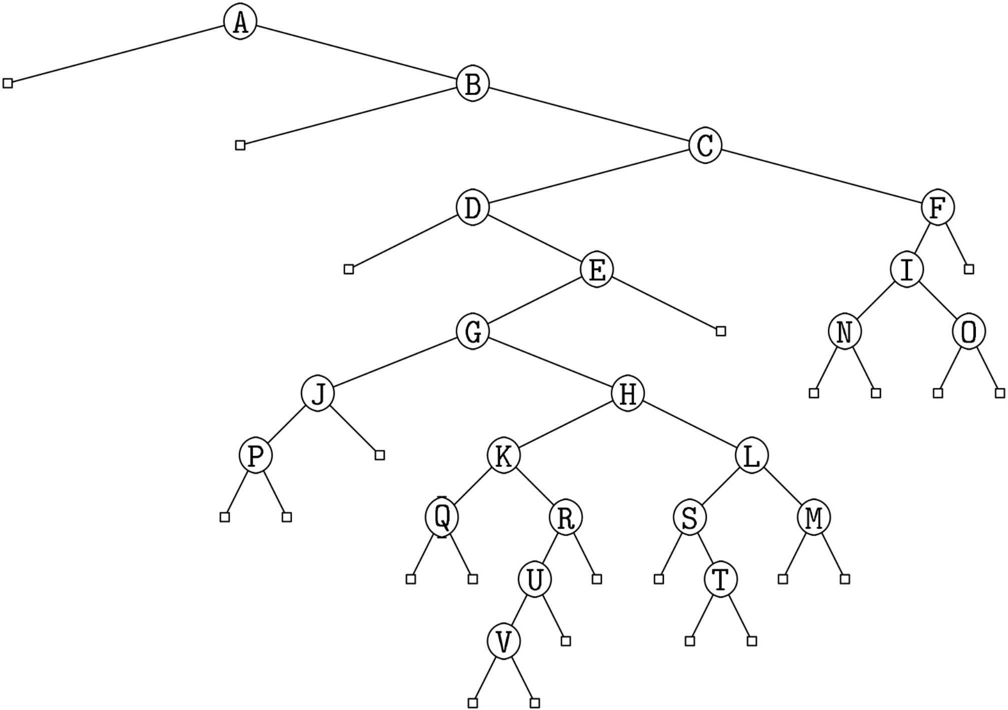
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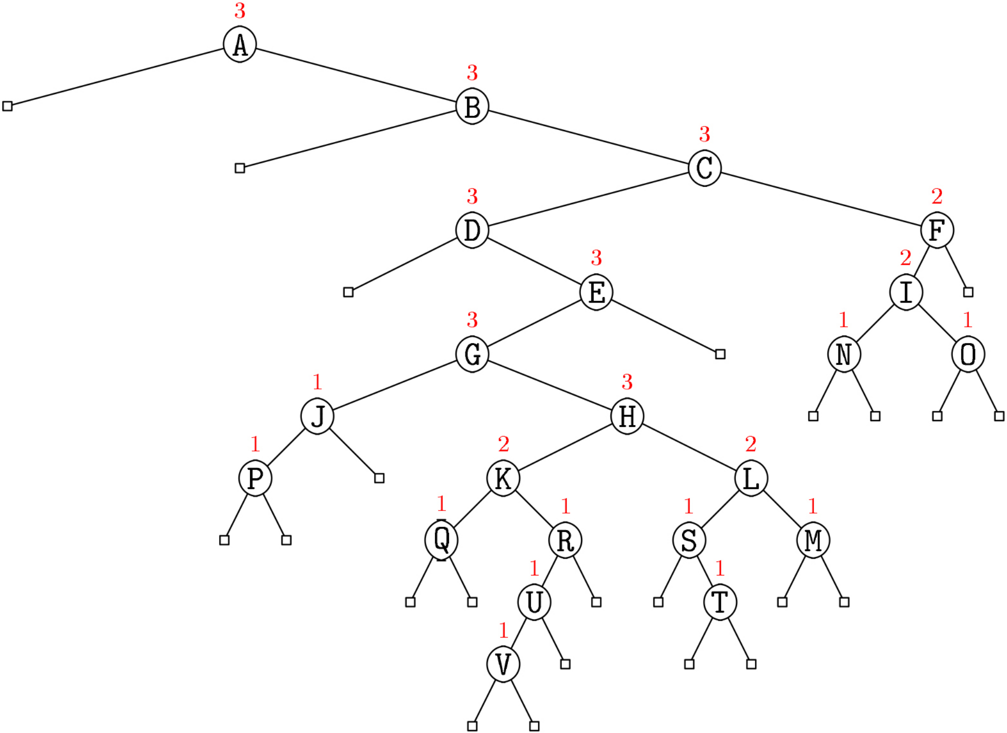
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11. Checking the Strahler number. If our implementation of Zeilberger's transformation is correct, it will have set $q[x]$ to the Strahler number of the binary subtree rooted at x with respect to the *ll* and *rr* links, for every node x.

Therefore we want to check this condition. And we might as well do the checking by brute force, so that the evidence is convincing.

12. ⟨ Subroutines 5 ⟩ +≡

```
int strahler(register int x)
{
  register int sl, sr, s;
  if (ll[x]) sl = strahler(ll[x]);
  else sl = 0;
  if (rr[x]) sr = strahler(rr[x]);
  else sr = 0;
  s = (sl > sr ? sl : sl < sr ? sr : sl + 1);
  if (q[x] ≠ s)
    fprintf(stderr, "I␣goofed␣at␣binary␣\
        tree␣node␣%d,␣case␣%d.\n", x,
        serial);
  return s;
}
```

R.A.I.R.O. Informatique théorique/Theoretical Informatics
(vol. 18, n° 4, 1984, p. 355 à 364)

SUR LE NOMBRE DE REGISTRES NÉCESSAIRES A L'ÉVALUATION D'UNE EXPRESSION ARITHMÉTIQUE (*)

par Jean FRANÇON (1)

Communiqué par R. CORI

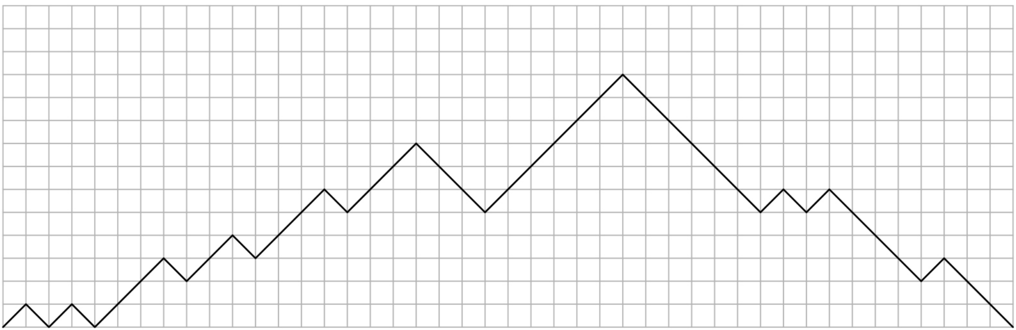
Résumé. — *La distribution de la variable « nombre de registres nécessaires à l'évaluation d'une expression arithmétique » sur les arbres binaires est étudiée de façon purement combinatoire. Une nouvelle relation de récurrence est démontrée. Cette relation permet une preuve nouvelle et sans calcul du théorème de Flajolet, Raoult et Vuillemin liant la distribution en question à celle de la « hauteur à gauche » d'un arbre binaire.*

Abstract. — *The distribution of the variable "number of registers required for evaluating an arithmetic expression" on binary trees is studied in a purely combinatorial way. A new recurrence relation is proved. This relation gets a new proof, without calculations, of the theorem of Flajolet, Raoult and Vuillemin linking this distribution to the distribution of the "left height" of a binary tree.*

1. INTRODUCTION

On sait qu'une expression arithmétique dont tous les opérateurs sont binaires est représentable par un arbre binaire (dans la terminologie de [10] section 2.3, p. 305, que nous utilisons dans cette note). On sait aussi, depuis les travaux d'Ershov [1] et de Sethi et Ullman [12], que le nombre minimum de registres de mémoire nécessaire à l'évaluation d'une expression arithmétique, dans les modèles courants de calcul, est une variable d'arbre binaire

2. And what is a nested string? A nested string (aka Dyck word) of order n is a sequence $d_0, d_1, \ldots, d_{2n-1}$ of ±1s whose partial sums $y_k = d_0 + \cdots + d_k$ are nonnegative, and whose overall sum y_{2n} is zero. Its height is $\max_{0 \le k < 2n} y_k$. The bijection implemented in this program associates the example tower above with the nested string having



as its graph of partial sums; in this case the height is 11.

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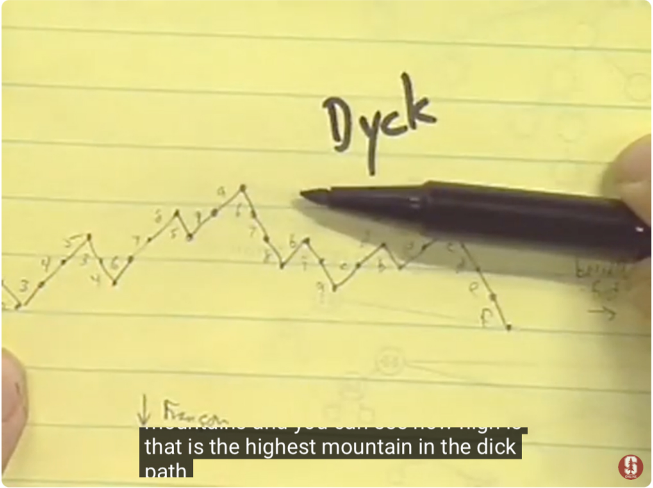

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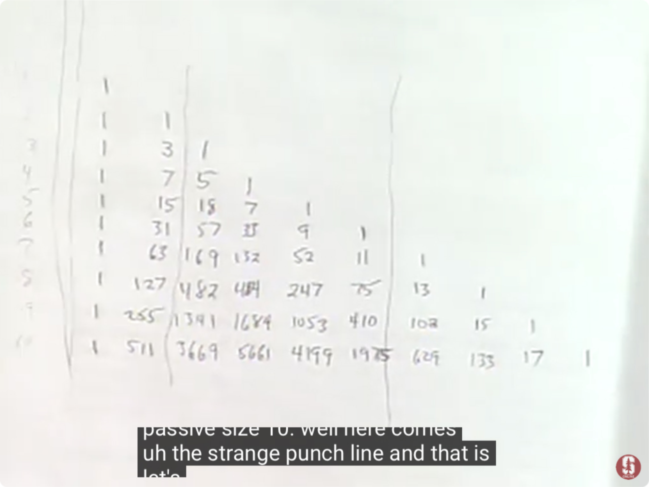

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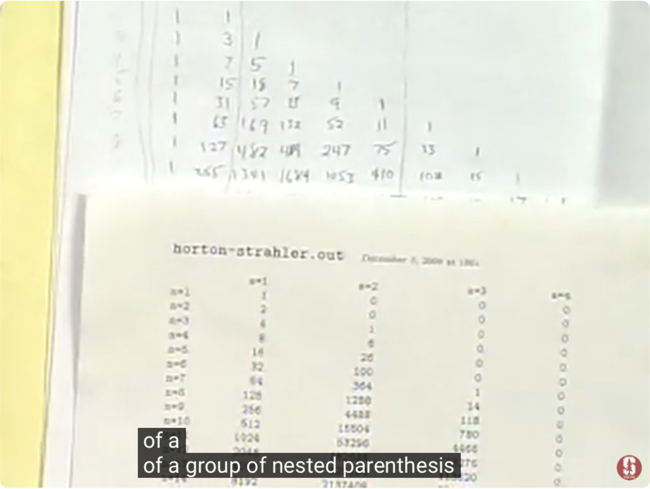

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En d'autres termes : notons B_n l'ensemble des arbres binaires de taille (nombre de sommets internes) n, pour tout entier $n \geqslant 0$; notons $R_{p,n}$ le nombre d'arbres $b \in B_n$ tels que $\operatorname{Reg}(b) = p$, pour tout entier $p \geqslant 0$; notons $H_{k,n}$ le nombre d'arbres $b \in B_n$ tels que $\operatorname{Hag}(b) = k$, k entier $\geqslant 0$; alors on a

$$R_{p,n} = \Sigma H_{k,n}$$

où la somme s'étend aux entiers k tels que $p = [\operatorname{Log}_2(1+k)]$. Nous appelons les entiers $R_{p,n}$ *nombres de Strahler*.

On peut constater ce fait sur les tables des premières valeurs des nombres $R_{p,n}$ et $H_{k,n}$ ci-dessous : la première colonne des $R_{p,n}$ est égale à la somme des deux premières colonnes des $H_{k,n}$ et la colonne $R_{2,n}$ est la somme des colonnes restantes, soit $H_{3,n}$, $H_{4,n}$ et $H_{5,n}$, puisque les tables sont limitées à $n = 5$.

TABLE 1.1.
$R_{p,n}$ (*tiré de* [5])

n \ p	1	2
1	1	
2	2	
3	4	1
4	8	6
5	16	26

TABLE 2.1.
$H_{k,n}$ (*par dénombrement direct*)

n \ k	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	5	1	
5	1	15	18	7	1

Ce théorème est surprenant. En effet, la variable Reg est évidemment invariante par échange des deux sous-arbres d'un arbre binaire, alors que la variable Hag ne l'est en général pas. Cette remarque nous a conduit à examiner de plus près la combinatoire des arbres binaires. L'objet de cette note est de montrer d'abord qu'une construction combinatoire tout à fait élémentaire permet de trouver une nouvelle relation de récurrence pour les nombres de Strahler (section 2), puis (section 3) de montrer que des constructions soit très classiques, soit élémentaires permettent d'obtenir la même relation de récurrence pour la distribution de la variable $[\operatorname{Log}_2(1+\operatorname{Hag})]$, fournissant ainsi une nouvelle preuve du théorème 1.1.

Construire une bijection (suffisamment simple) de l'ensemble des arbres binaires de taille donnée sur lui-même transformant la variable Reg en la variable $[\operatorname{Log}_2(1+\operatorname{Hag})]$ reste toutefois un problème ouvert.

On the Generation of Binary Trees

ANDRZEJ PROSKUROWSKI

University of Oregon, Eugene, Oregon

ABSTRACT. A binary tree may be uniquely represented by a *code* reflecting traversal of the corresponding *extended binary tree* in a given *monotonic* order. A general algorithm for constructing codes of all binary trees with *n* vertices is presented. Different orders of traversal yield different orderings of the generated trees. The algorithm is illustrated with an example of the sequence of binary trees obtained from ballot sequences.

KEY WORDS AND PHRASES: code, binary tree, traversal, ordering, generation

CR CATEGORIES: 5.31, 5.39

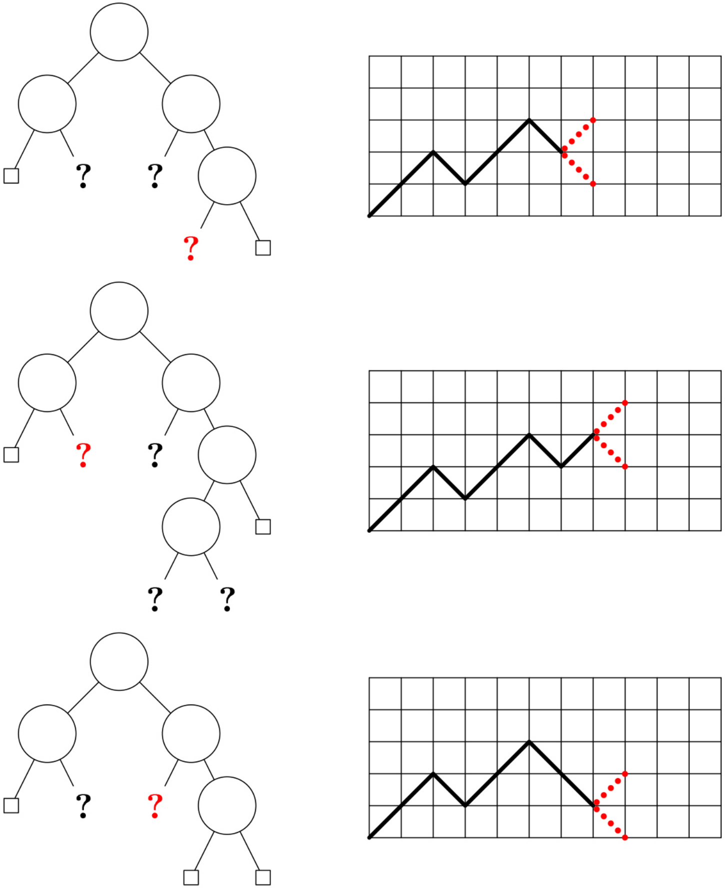
The problem of constructing all binary trees with *n* vertices is equivalent to that of generating all *extended binary trees* ([2], 2.3.4.5) with $n + 1$ *leaves*, where all *internal vertices* have both left and right sons. The correspondence between binary trees with *n* vertices and extended binary trees with $n + 1$ leaves is obviously bijective. The algorithm constructing all extended binary trees is based on a procedure for "expanding" certain leaves of trees with *k* leaves to obtain trees with $k + 1$ leaves. In order to avoid generation of isomorphic trees, this expansion is controlled by an ordering of the vertices of a tree. We define *monotonic* orderings as such orderings which place descendants after their ancestors. Preorder traversal and level order are examples of such orderings. In the procedure **expand** presented below, the ordering relation (corresponding to a monotonic ordering of vertices) is denoted "<" and the tree representation is being assembled in a global structure *tree*.

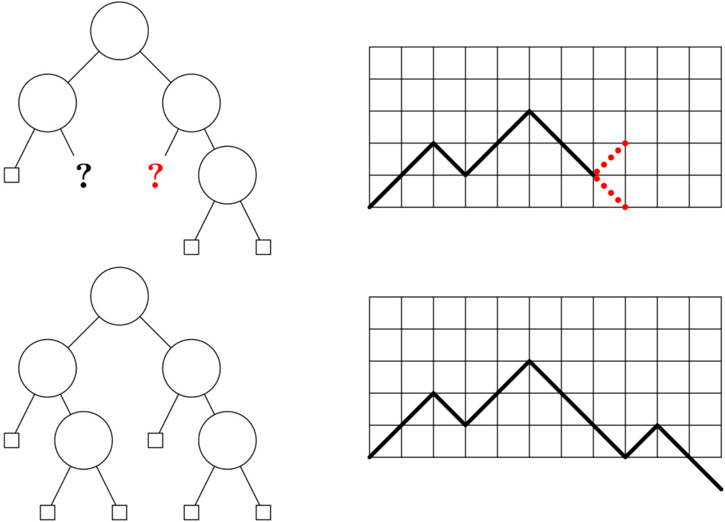
```
procedure expand(a: leaf);
begin  augment tree by a;
       if more vertices needed
       then begin order vertices of tree;
                for each leaf l > a in order
                     expand(l)
            end
       else output tree
       end;
```

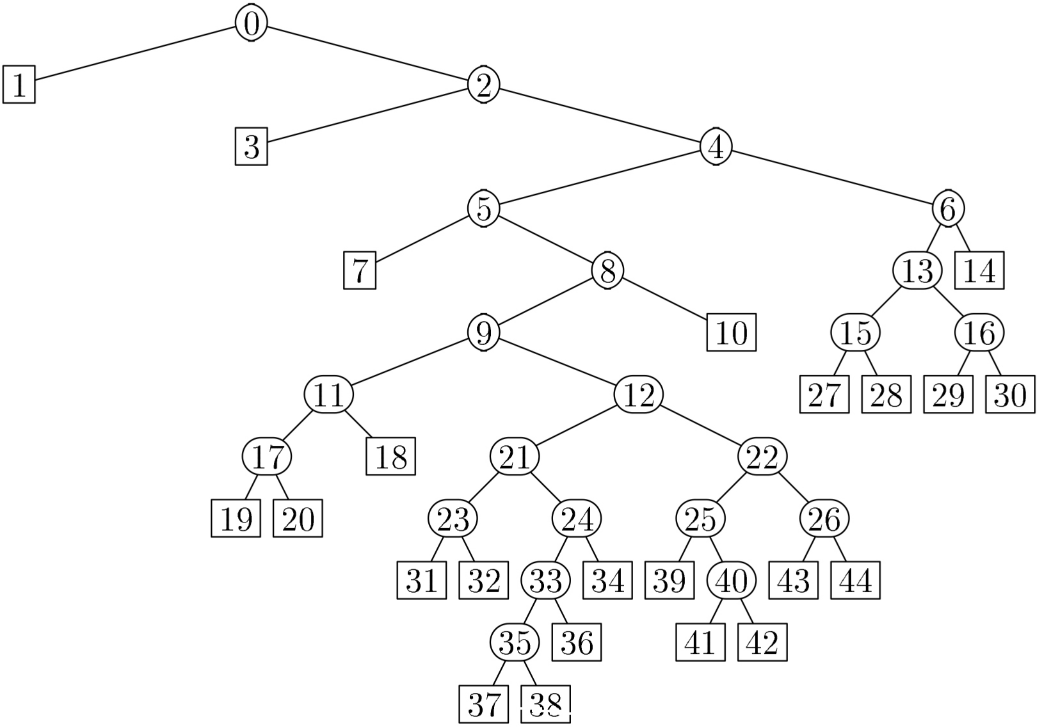
The above procedure promoting leaves to the status of internal vertices by "expanding" them is based on the following observation. Due to the monotonic property of the ordering controlling the construction process, vertices of any finite binary tree are ordered according to the ordering of the corresponding vertices of the complete infinite binary tree. Procedure **expand** simply lists all binary trees with *n* vertices (represented by lists of vertices, most recently expanded last) in lexicographical order defined by the ordering of vertices.

We will show that our procedure is an *orderly algorithm* [4] and thus produces every binary tree of *n* vertices exactly once. We will represent an extended binary tree by its binary *code* depending on the chosen ordering. The *k*th bit of the code of a tree *T* with

Author's address: Department of Computer Science, University of Oregon, Eugene, OR 97403.



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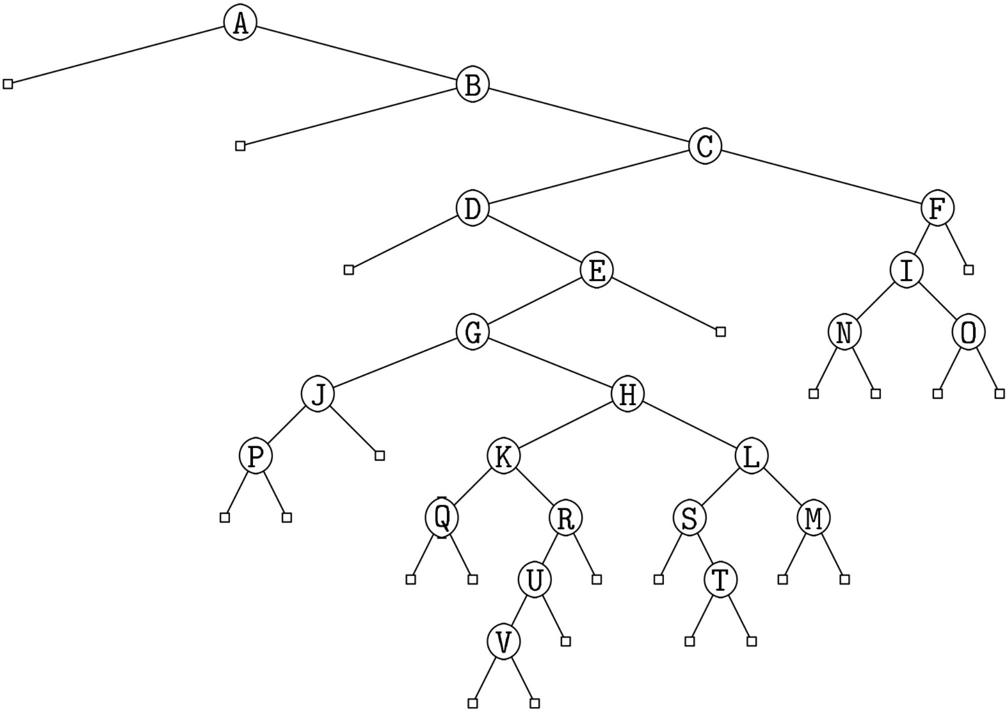
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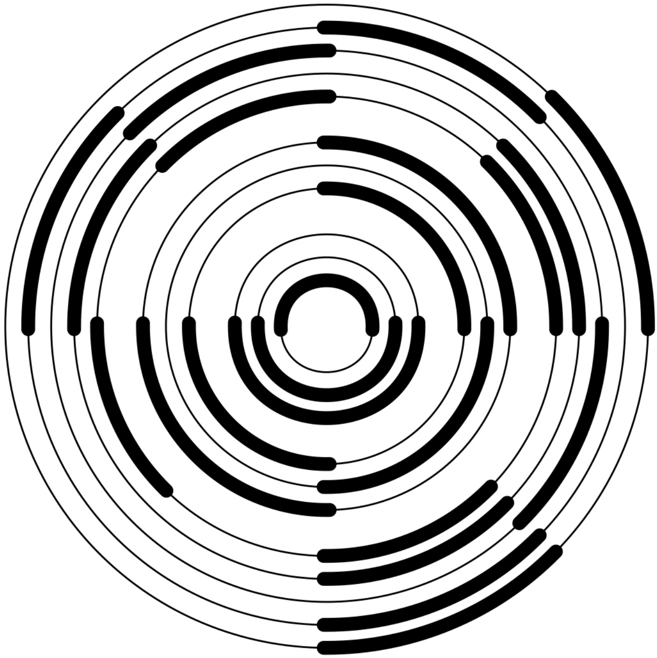
76. *Kepler towers* with *n bricks*, i.e., sets of concentric circles, with "bricks" (arcs) placed on each circle, as follows: the circles come in sets called *walls* from the center outward. The circles (or *rings*) of the ith wall are divided into 2^i equal arcs, numbered $1,2,\ldots,2^i$ clockwise from due north. Each brick covers an arc and extends slightly beyond the endpoints of the arc. No two consecutive arcs can be covered by bricks. The first (innermost) arc within each wall has bricks at positions $1,3,5,\ldots,2^i-1$. Within each wall, each brick B not on the innermost ring must be supported by another brick B' on the next ring toward the center, i.e., some ray from the center must intersect both B and B'. Finally, if $i>1$ and the ith wall is nonempty, then wall $i-1$ must also be nonempty. Figure 2.1 shows a Kepler tower with three walls, six rings, and thirteen bricks.

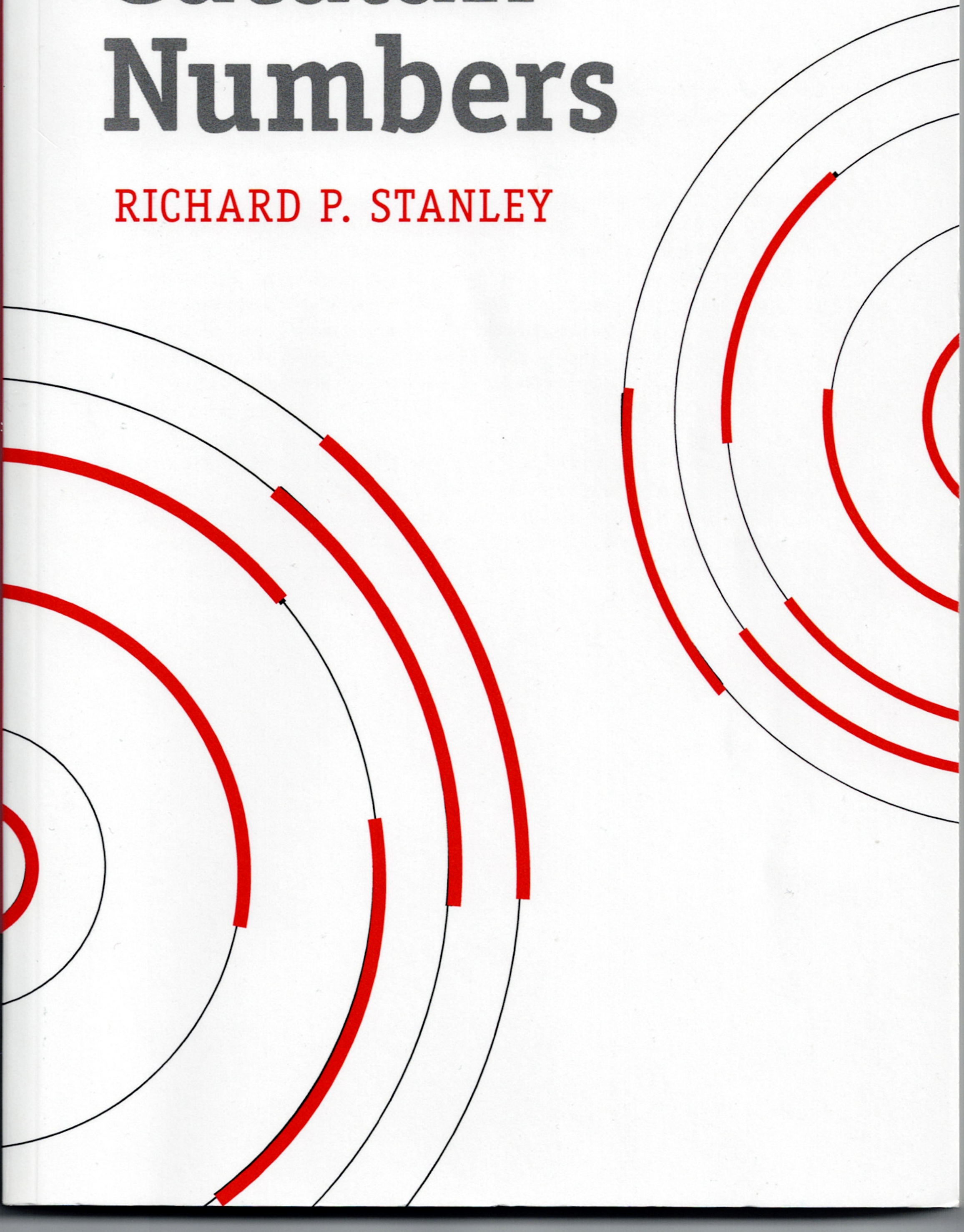
What is a Kepler tower? Good question. It is a new kind of combinatorial object, invented by Xavier Viennot in February 2005. For example,

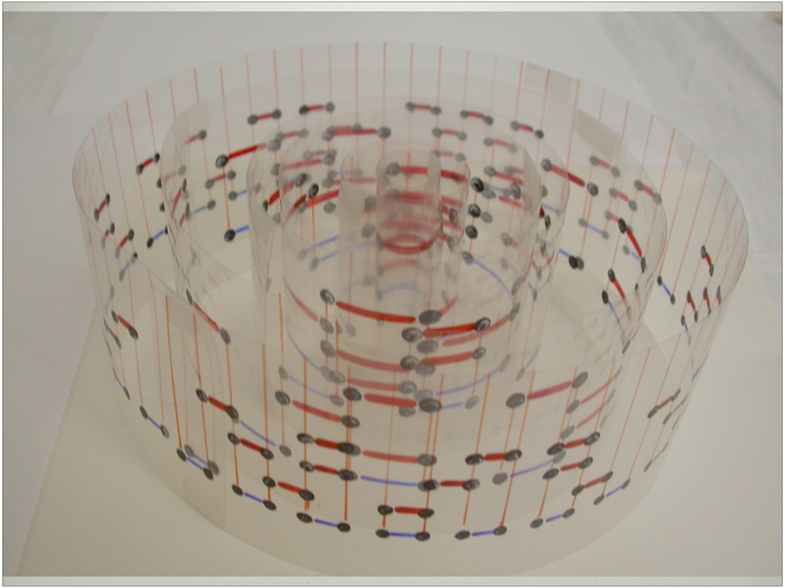


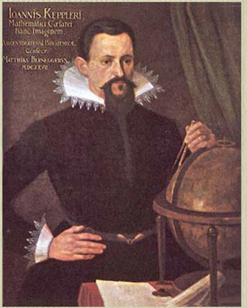
depicts a Kepler tower with 3 walls containing 22 bricks. This illustration is two-dimensional, but of course a tower has three dimensions; we are viewing the tower from above. Every wall of a Kepler tower consists of one or more rings, where each ring of the kth wall is divided into 2^k segments of equal length. Each brick is slightly longer than the length of one segment.

Catalan Numbers

RICHARD P. STANLEY







Mysterium cosmographicum (1596)

BACK-KEPLER-TOWERS

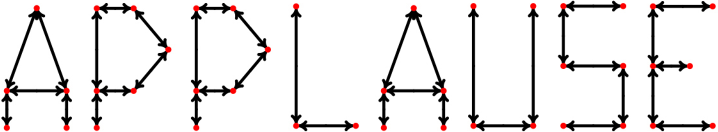
1. Intro. This program generates all Kepler towers made from n bricks. (It supplements the old program VIENNOT in my Mittag-Leffler report "Three Catalan bijections," which was incomplete: The claim that all towers are generated was never proved, because I'd blithely assumed that there are no more than C_n of them.)

```
#define maxn 40    /* this is plenty big, since C_40 > 10^21 */

#include <stdio.h>
#include <stdlib.h>
  int n;    /* command line parameter */
  int x[maxn];    /* current brick position */
  int w[maxn];    /* current wall number */
  int p[maxn];
    /* beginning of supporting layer */
  int q[maxn];
    /* beginning of current layer */
  int t[maxn];    /* type of move: 1 if end of
      layer, 2 if end of wall */
  char punct[3] = {',', ';', ':'};
    /* separators */
  unsigned long long count;
    /* this many found */

  main(int argc, char *argv[])
  {
    register i, j, k, l, mask;

    ⟨ Process the command line 2 ⟩;
  b1: ⟨ Initialize for backtracking 4 ⟩;
```


APPLAUSE