Some problems I have proposed

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1. **Asian Pacific Mathematics Olympiad 2017, Problem 1**

   We call a 5-tuple of integers *arrangeable* if its elements can be labeled $a, b, c, d, e$ in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, \ldots, n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

2. **Asian Pacific Mathematics Olympiad 2017, Problem 5**

   (with Pakawut Jiradilok)

   Let $n$ be a positive integer. A pair of $n$-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ with integer entries is called an *exquisite pair* if
   
   $$|a_1b_1 + \cdots + a_nb_n| \leq 1.$$  

   Determine the maximum number of distinct $n$-tuples with integer entries such that any two of them form an exquisite pair.

3. **International Mathematical Olympiad 2016 Shortlist, Problem N1**

   For any positive integer $k$, denote the sum of digits of $k$ in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and
   
   $$S(P(n)) = P(S(n)).$$

4. **Asian Pacific Mathematics Olympiad 2016, Problem 3**

   Let $AB$ and $AC$ be two distinct rays not lying on the same line, and let $\omega$ be a circle with center $O$ that is tangent to ray $AC$ at $E$ and ray $AB$ at $F$. Let $R$ be a point on segment $EF$. The line through $O$ parallel to $EF$ intersects the line $AB$ at $P$. Let $N$ be the intersection of lines $PR$ and $AC$, and let $M$ be the intersection of line $AB$ and the line through $R$ parallel to $AC$. Prove that line $MN$ is tangent to $\omega$.

5. **Asian Pacific Mathematics Olympiad 2016, Problem 4**

   The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest positive integer $k$ such that no
matter how Starways establishes its flights, the cities can always be partitioned into \( k \) groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

6. **Asian Pacific Mathematics Olympiad 2015, Problem 1**

Let \( ABC \) be a triangle, and let \( D \) be a point on side \( BC \). A line through \( D \) intersects side \( AB \) at \( X \) and ray \( AC \) at \( Y \). The circumcircle of triangle \( BXD \) intersects the circumcircle \( \omega \) of triangle \( ABC \) again at point \( Z \neq B \). The lines \( ZD \) and \( ZY \) intersect \( \omega \) again at \( V \) and \( W \), respectively. Prove that \( AB = VW \).

7. **Asian Pacific Mathematics Olympiad 2015, Problem 4**

(with Pakawut Jiradilok)

Let \( n \) be a positive integer. Consider \( 2n \) distinct lines on the plane, no two of which are parallel. Of the \( 2n \) lines, \( n \) are colored blue, the other \( n \) are colored red. Let \( B \) be the set of all points on the plane that lie on at least one blue line, and \( R \) the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects \( B \) in exactly \( 2n - 1 \) points, and also intersects \( R \) in exactly \( 2n - 1 \) points.

8. **Asian Pacific Mathematics Olympiad 2015, Problem 5**

(with Pakawut Jiradilok)

Determine all sequences \( a_0, a_1, a_2, \ldots \) of positive integers with \( a_0 \geq 2015 \) such that for all integers \( n \geq 1 \):

(i) \( a_{n+2} \) is divisible by \( a_n \);

(ii) \( |s_{n+1} - (n + 1)a_n| = 1 \), where \( s_{n+1} = a_{n+1} - a_n + a_{n-1} - \cdots + (-1)^{n+1}a_0 \).

9. **Asian Pacific Mathematics Olympiad 2014, Problem 2**

Let \( S = \{1, 2, \ldots, 2014\} \). For each non-empty subset \( T \subseteq S \), one of its members is chosen as its representative. Find the number of ways to assign representatives to all non-empty subsets of \( S \) so that if a subset \( D \subseteq S \) is a disjoint union of non-empty subsets \( A, B, C \subseteq S \), then the representative of \( D \) is also the representative of at least one of \( A, B, C \).

10. **Asian Pacific Mathematics Olympiad 2014, Problem 3**

Find all positive integers \( n \) such that for any integer \( k \) there exists an integer \( a \) for which \( a^3 + a - k \) is divisible by \( n \).

11. **International Mathematical Olympiad 2013, Problem 4**

(with Potcharapol Suteparuk)

Let \( ABC \) be an acute-angled triangle with orthocenter \( H \), and let \( W \) be a point on the side \( BC \), lying strictly between \( B \) and \( C \). The points \( M \) and \( N \) are the feet of the altitudes from \( B \) and \( C \), respectively. Denote by \( \omega_1 \) the circumcircle of \( BWN \), and let \( X \) be the point on \( \omega_1 \) such that \( WX \) is a diameter of \( \omega_1 \). Analogously,
denote by $\omega_2$ the circumcircle of $CWM$, and let $Y$ be the point on $\omega_2$ such that $WY$ is a diameter of $\omega_2$. Prove that $X, Y$ and $H$ are collinear.

12. United States of America Mathematical Olympiad 2013, Problem 3

Let $n$ be a positive integer. There are $n\frac{(n+1)}{2}$ marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing $n$ marks. Initially, each mark has the black side up. An operation is to choose a line parallel to one of the sides of the triangle, and flipping all the marks on that line. A configuration is called admissible if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration $C$, let $f(C)$ denote the smallest number of operations required to obtain $C$ from the initial configuration. Find the maximum value of $f(C)$, where $C$ varies over all admissible configurations.

13. International Mathematical Olympiad 2012 Shortlist, Problem C1

Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers $x$ and $y$ such that $x > y$ and $x$ is to the left of $y$, and replaces the pair $(x, y)$ by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

14. International Mathematical Olympiad 2012 Shortlist, Problem N1

Call admissible a set $A$ of integers that has the following property:

If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer $k$.

Determine all pairs $m, n$ of nonzero integers such that the only admissible set containing both $m$ and $n$ is the set of all integers.

15. United States of America Junior Mathematical Olympiad 2012, Problem 5

For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers $k$ with $1 \leq k < 2012$ such that the remainder when $ak$ divided by 2012 is greater than that of $bk$ divided by 2012. Let $S$ be the minimum value of $f(a, b)$, where $a$ and $b$ range over all pairs of distinct positive integers less than 2012. Determine $S$.

16. International Mathematical Olympiad 2011 Shortlist, Problem A2

Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers such that for every positive integer $n$ there is an integer $a$ with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$