Verifying Stochastic Systems

Jacob Steinhardt and Russ Tedrake

Massachusetts Institute of Technology
Motivation

- Robots are often subject to large uncertainties
  - dynamical: wind gusts
  - perceptual: stereo vision
- To maximize performance, want to plan against typical case (99%) rather than worst case (100%)
Lyapunov equations
Lyapunov equations

• Sufficient condition for stability: non-negative function $V$ such that

$$\dot{V}(x) \leq 0$$
Martingale condition
Martingale condition

\[ \mathbb{E}[\dot{V}(x)] \leq 0 \]
Martingale condition

\[ \mathbb{E}[\dot{V}(x)] \leq 0 \]

\[ \lim_{\Delta t \to 0^+} \frac{\mathbb{E}[V(x(t + \Delta t)) \mid x(t)] - V(x(t))}{\Delta t} \]
Martingale condition
Martingale condition

$$E[\dot{V}(x)] \leq 0$$

• Too strong!
Martingale condition

$$\mathbb{E}[\dot{V}(x)] \leq 0$$

• Too strong!
• Consider the system
Martingale condition

$$E[\dot{V}(x)] \leq 0$$

• Too strong!
• Consider the system
Martingale condition

\[ \mathbb{E}[\dot{V}(x)] \leq 0 \]

• Too strong!

• Consider the system

\[ dx(t) = -x dt + dw(t) \]

• What is \( \mathbb{E}[\dot{V}(0)] \)?
Martingale condition
Martingale condition

• Instead consider the relaxed condition

\[ \mathbb{E}[\dot{V}(x)] \leq c \]
Martingale condition

• Instead consider the relaxed condition

\[ \mathbb{E}[\dot{V}(x)] \leq c \]

• theorems giving finite-time guarantees (Kushner 1965)
Martingale condition

- Instead consider the relaxed condition
  \[ \mathbb{E}[\dot{V}(x)] \leq c \]

- theorems giving finite-time guarantees (Kushner 1965)

- todo: choose and optimize over a family of functions \( V \)
Choosing V
Choosing $V$

• $V(x) = \exp(x^T J x)$
Choosing $V$

- $V(x) = \exp(x^T J x)$
- **NB:** using $x^T J x$ instead has poor results (no bound at all after a few seconds)
Choosing V

• $V(x) = \exp(x^T J x)$

• NB: using $x^T J x$ instead has poor results (no bound at all after a few seconds)

• $E[\frac{dV}{dt}] = p(x) \exp(x^T J x)$
Verifying the Martingale Condition
Verifying the Martingale Condition

• from previous slide: $\mathbb{E}[\dot{V}(x)] = p(x)e^{x^TJx}$
Verifying the Martingale Condition

• from previous slide: $\mathbb{E}[\dot{V}(x)] = p(x)e^{x^T J x}$

\[ p(x)e^{x^T J x} \leq c \]

\[ \iff p(x) \leq ce^{-x^T J x} \]

\[ \iff p(x) \leq c(1 - x^T J x) \]
Verifying the Martingale Condition

• from previous slide: \( \mathbb{E}[\dot{V}(x)] = p(x)e^{x^T Jx} \)

\[
p(x)e^{x^T Jx} \leq c
\]

\[
\iff p(x) \leq ce^{-x^T Jx}
\]

\[
\iff p(x) \leq c(1 - x^T Jx)
\]

• polynomial condition: optimize with SOS programming
Results: Quadrotor
Results: UAV
Further reading

- http://groups.csail.mit.edu/locomotion/
- LQR-trees tutorial on Friday in the “Integrated Planning and Control” workshop