Redundancy

Equations of Motion
Joint Space
\[ A(q)\ddot{q} + b(q, \dot{q}) + g(q) = \Gamma \]
Operational Space
\[ \Lambda(x)\ddot{x} + \mu(x, \dot{x}) + p(x) = F \]
Relationships
\[ \Gamma = J^T F \]
\[ (A\ddot{q} + b + g) = J^T (\Lambda\ddot{x} + \mu + p) \]
\[ (A\ddot{q} + b) = J^T (\Lambda\ddot{x} + \mu) \]
Inertial forces

Non Redundancy
\[ A\ddot{q} + b + g = \Gamma \] (joint dynamics)
\[ J^{-T} \]
\[ J^T \]
\[ \Lambda\ddot{x} + \mu + p = F \] (Task dynamics)
Redundancy

\[ A \ddot{q} + b + g = \Gamma \]  
(joint dynamics)

\[ J^T \]

projection

\[ \Lambda \ddot{x} + \mu + p = F \]  
(Task dynamics)

where

\[ \bar{J} = A^{-1}J^T \Lambda \quad \text{and} \quad \Lambda^{-1} = J A^{-1}J^T \]

\( \bar{J} \): dynamically consistent generalized inverse

Redundancy

Joint/Task Displacements

\[ \delta x = J \delta q \]

\[ \delta q = J^* \delta x + \left[ I - J^* J \right] \delta q_0 \]

Joint/Task Forces

\( \Gamma = J^T F \)

\( F = ? \)

However

different selections of \( J^* \) \( (J = J J^* J) \)

would lead
to different solutions

Redundancy

\[ \delta q = J^* \delta x + \left[ I - J^* J \right] \delta q_0 \]

\( \Gamma = J^T F \)

Given \( F \), \( \Gamma \) is \( (J^T F) \)

Given \( \Gamma \), what is \( F \)?

\( F = J^* \Gamma \) ?

Gravity Example

\[ p = ? \]

\[ p = J^* g \]

\[ J^+ = J^T \left( J J^T \right)^{-1} \]
Gravity Example

\[ p = ? \]
\[ p = J^T g \]
\[ J = A^{-1} J^T \Lambda \]

Redundancy

\[ \delta q = J^w x + \left( I - J^w J \right) \delta q_0 \]
\[ \delta w = \Gamma^T \delta q \]
\[ \delta w = \delta w_1 + \delta w_2 \]
\[ \left( J^w \Gamma \right)^T \delta x \]
\[ \left( I - J^w J \right)^T \Gamma \delta q_0 \]
\[ \Gamma = J^T \left( J^w \Gamma \right) + \left( I - J^T J^w \right) \Gamma \]

Decomposition

\[ \Gamma = J^T \left[ J^w \Gamma \right] + \left[ I - J^T J^w \right] \Gamma \]

Virtual Displacement

\[ \delta q = J^w x + \left( I - J^w J \right) \delta q_0 \]

Virtual Work

\[ \delta w = \Gamma^T \delta q \]

Dynamic Constraints

\[ A(q) \ddot{q} + b(q, \dot{q}) + g(q) = \Gamma \]
\[ \Gamma = J^T F + \left[ I - J^T J^w \right] \Gamma_0 \]
\[ A \ddot{q} + (b + g) = J^T F + \left[ I - J^T J^w \right] \Gamma_0 \]

\[ \ddot{q} + A^{-1} \left( b + g \right) = A^{-1} J^T F + A^{-1} \left[ I - J^T J^w \right] \Gamma_0 \]
\[ J \dddot{q} + JA^{-1} \left( b + g \right) = JA^{-1} J^T F + JA^{-1} \left[ I - J^T J^w \right] \Gamma_0 \]
\[ J \dddot{q} = \dddot{x} - J \dddot{q} \]
\[ \dddot{x} + \left[ JA^{-1} \left( b + g \right) - J \dddot{q} \right] = \left( JA^{-1} J^T \right) F + JA^{-1} \left[ I - J^T J^w \right] \Gamma_0 \]
\[ \Lambda^{-1} \dddot{x}_n = 0 \]
Dynamic Consistency

\[ \Gamma = J^T \Gamma + \left( I - J^T J \right) \Gamma_0 \]

Dynamic Constraint

\[ JA^{-1} \left( I - J^T J \right) \Gamma_0 = 0 \]

\[ \Lambda^{-1} \left( JA^{-1} J^T \right) -1 JA^{-1} = J^{\#} \]

\[ \tilde{J}(q) \] is the Dynamically Consistent Generalized Inverse

Theorem (Consistency)

\[ \tilde{J} \] is unique and \[ \tilde{J} = A^{-1} J^T \Lambda \]

Non-redundant

\[ \tilde{J} = J^{-1} \]

Velocity Force Duality

\[ \delta q = J^{-1} \delta x \]

Non Red.

\[ \delta q = \tilde{J} \delta x + \left[ I - \tilde{J} J \right] \delta q_0 \]

\[ \Gamma = J^T \Gamma \]

\[ \Gamma = J^T F \]

\[ \delta q = J^T F + \left[ I - J^T J \right] \Gamma_0 \]

Task dynamics

\[ \Lambda(q) \ddot{x} + \mu(q, \dot{q}) + p(q) = F \]

\[ \Lambda = \left( JA^{-1} J^T \right)^{-1} \]

\[ \mu(q, \dot{q}) = \tilde{J}^T b(q, \dot{q}) - \Lambda(q) \tilde{J}(q) \dot{q} \]

\[ p(q) = \tilde{J}^T g(q) \]
Redundant Robot Control

Task Space: \( J^T \)
Null Space: \( N^T \) where \( N = I - JJ \)

Robot Control
\[
\Gamma = J^T F + N^T \Gamma_0
\]
\( \Gamma_1 \) \quad \( \Gamma_2 \)
dynamically decoupled

\( J^T J \): is a \( n \times n \) matrix of rank \( m_0 \)
it is Positive Semi-definite

The System is Stable, but not asymptotically stable
\[
\dot{q}^T D(q) \dot{q} = 0
\]

Stability
\[
\Gamma_{dis}^T \dot{q} \leq 0 \quad \text{for} \quad \dot{q} \neq 0
\]
\[
\Gamma_{dis} = -k_v J^T \dot{x} = -k_v J^T J \dot{q}
\]
\[
\dot{q}^T D(q) \dot{q} \geq 0 \quad ; \quad \dot{q} \neq 0
\]
\[
D(q) = k_v \left( J^T J \right)
\]

Asymptotic Stability
\[
\Gamma_{dis}^T \dot{q} < 0 \quad ; \quad \text{for} \quad \dot{q} \neq 0
\]
\[
\Gamma_{dis} = -k_v J^T J \dot{q} - k_v N^T \dot{q}
\]
\[
\Downarrow
\]
\[
D(q) = k_v \left( J^T J + N^T \right)
\]
Positive definite
\[
\dot{q}^T D(q) \dot{q} < 0 \quad \text{for} \quad \dot{q} \neq 0
\]
Kinematic Singularities

Joint Space Formulation
Find a pseudo-inverse $J^+$

Pseudo Inverse Solution

$$\Delta q_1 = \frac{l_1 + l_2}{l_2^2 (l_1 + l_2)^2} \delta y_1 \rightarrow \theta_2$$

Kinematic Singularities

The end-effector mobility locally decreases

Singularities
$S(q) = \det[J(q)] = S_1(q).S_2(q) \cdots S_n(q)$

Singular direction
$S_i = 0$ \hspace{0.5cm} Infinite effective mass
$\zeta_i$ \hspace{0.5cm} Infinite effective inertia

Singularity Neighborhood

$S(q) = S_1(q).S_2(q).S_3(q) \cdots S_n(q)$

Singularity $S_i$
$D_{S_i} = \{ q \mid |S_i(q)| \leq S_0 \}$

Singularity Neighborhood

$|S| \leq S_0$

Singularity Neighborhood

Approach

In $D_{S_i}$, the robot is treated as redundant w.r.t. motions in the subspace \perp to the singular direction

- Along Singular Directions:
  - Control in Null Space $\Gamma_{null-space}$

- In subspace \perp to singular direction
  - Control in sub-O-Space $F_{sub-os}$
Types of Singularities

Elbow Lock  Wrist Lock  Overhead Lock
Type 1  Type 2

Types of Singularities

Control Strategy
Type 1
Motion in Null Space
⇒ Motion along/about $\zeta_i$
Control $S_i$

Type 2
Motion in Null Space
⇒ Only changes of $\zeta_i$
Control $\zeta_i$

Singularity Control

$$\Gamma = J_{\text{sub}}^T F_{\text{sub}} + N_{\text{sub}}^T \Gamma_s$$

where

$$N_{\text{sub}} = I - J_{\text{sub}} J_{\text{sub}}^T$$ and $$\Gamma_s = -\nabla V_i(S_i)$$

Moving to a singularity
Control $S_i(q)$ to reach $S_i = 0$

Moving out of a singularity
Control $\dot{S}_i$ from zero to the desired Velocity at the singularity boundary