

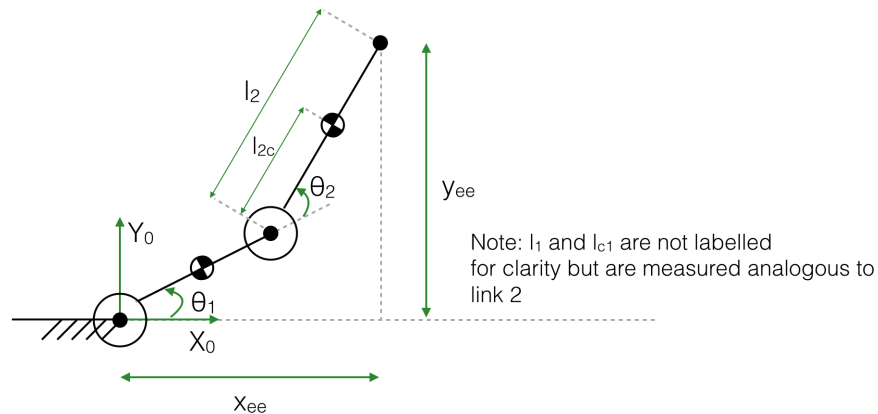
Solution Set #3

Problem 1 - Inertial properties

In this problem, you will explore the inertial properties of a manipulator at its end-effector.

(a) Design optimization of RR manipulator

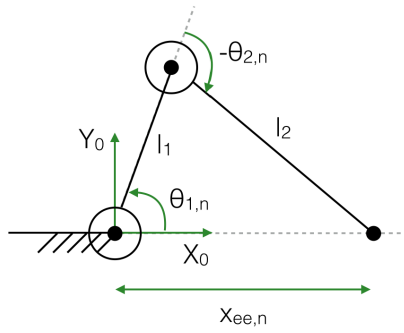
Suppose you have to design a RR planar manipulator as shown below to operate with an isotropic effective mass in the neighborhood of a given operating point. The only design parameters you are free to choose are the two link lengths, within some given range. Each link is modeled as a uniform cylinder with a fixed radius, $R = 0.1m$.



Answer the following questions:

- For the given nominal position of the end-effector, $x_{ee,n} = 0.75m$, $y_{ee,n} = 0.0m$, find the inverse kinematics solution in the elbow up position. Your solution, q_n must be a function of l_1 and l_2 .

Solution: We use the cosine law to find the required inverse kinematic solution.



$$\theta_{0,n} = \cos^{-1} \left(\frac{l_1^2 + x_{ee,n}^2 - l_2^2}{2l_1 x_{ee,n}} \right)$$

$$\theta_{1,n} = -\theta_{0,n} - \cos^{-1} \left(\frac{l_2^2 + x_{ee,n}^2 - l_1^2}{2l_2 x_{ee,n}} \right)$$

where the range of $b = \cos^{-1}(a)$ is assumed to be $b \in [0, \pi]$. Ofcourse, the solution exists only when $|a| \leq 1$.

- ii. Given $A(q)$ and $J_v(q)$ at the end-effector, compute the minimum and maximum effective mass at the end-effector for the configuration found above, $q = q_n$. Hint: Assume you are provided a function `sigma = svd(L)` that returns an array of singular values of the matrix L , $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ such that $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$. You can leave your answer in terms of the σ 's.

Solution: The effective mass in a direction u is given by

$$m_u = \frac{1}{u^T \Lambda_v^{-1} u}$$

So, the maximum value occurs when the quadratic form in the denominator is minimized. Similarly, it reaches its minimum when the denominator is at its maximum. Since Λ_v^{-1} is positive semi-definite, its singular value decomposition is given by

$$\Lambda_v^{-1} = Q S Q^T$$

where $Q \in \mathbb{R}^{2 \times 2}$ is a rotation matrix whose columns comprise of the eigenvectors of Λ_v^{-1} , and S is a diagonal matrix of the form

$$S = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

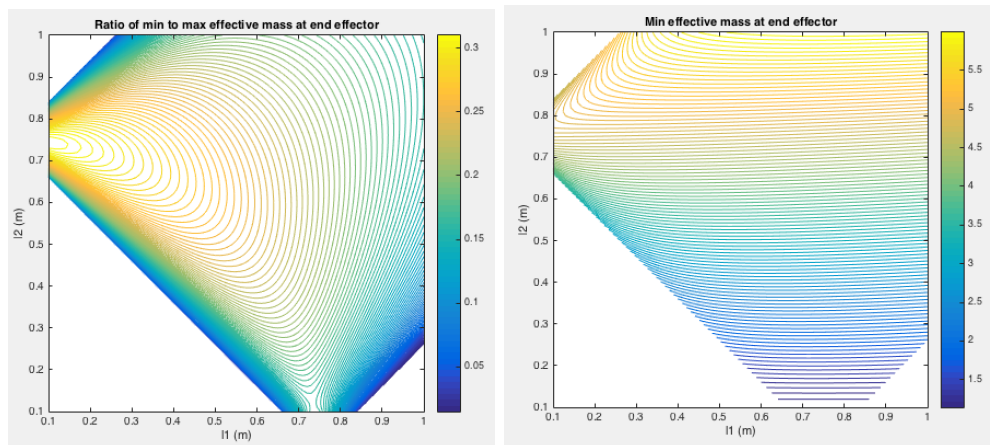
such that $\sigma_1 \geq \sigma_2 \geq 0$. Thus, the maximum value of the denominator occurs when $u = q_1$, the first column of Q , with a value of σ_1 . Likewise, the minimum value of the denominator occurs when $u = q_2$, the first column of Q , with a value of σ_2 . Thus,

$$m_{u,min} = \frac{1}{\sigma_1}, \quad m_{u,max} = \frac{1}{\sigma_2}$$

- iii. Over a range of $l_1 \in [0.1, 1.0]m$ and $l_2 \in [0.1, 1.0]m$, find a configuration of the manipulator at which the ratio of minimum to maximum effective mass is closest to unity. The MATLAB

script under `hw3/p1.m` is meant to assist you in this. Implement your solutions from the parts above in the portions marked as `FILL ME IN`. Running the script generates two contour plots, one showing the ratio of minimum to maximum singular values, and the other showing the minimum singular values across the different possible combinations of l_1 and l_2 . Attach the resulting plots. Explain the plots qualitatively.

Solution: The solution code for this problem can be found under `cs327a/hw3_sol/p1_sol.m`. Running it will generate two plots as follows:



Visibly, over the chosen range of l_1 and l_2 , the highest minimum to maximum mass ratio achieved is roughly 0.3, which occurs at about $l_1 = 0.1m$, $l_2 = 0.75m$. Here's a few things to observe about the two plots:

- On the ratio plot, the edges of the filled solution region indicate the boundaries of the feasible solutions given by the equations $l_1 + l_2 = 0.75$, $l_1 - l_2 = 0.75$ and $l_2 - l_1 = 0.75$. Along these lines, the manipulator is singular, and at feasible configurations close to these boundaries, the manipulator is almost singular.
- On the minimum effective mass plot, we see that for a particular values of l_2 , the minimum effective mass in any direction is largely independent of l_1 . This is due to the macro-mini inertial properties where we are guaranteed that the minimum effective mass is at most as large as that of the mini-manipulator alone. In this case, increasing l_1 can only increase the effective mass, but by a bounded amount.
- The two plots together suggest that there exists a trade-off in the design of this manipulator between the mass isotropy at the end-effector for the specified nominal workspace, and the minimum effective mass. Of course, in a real manipulator design problem, there are a lot more constraints such as the required workspace around the nominal work point, contact force isotropy, mechanical/electrical constraints etc.

(b) Belted ellipsoid of effective mass

For the RR manipulator above, now assume $l_1 = 0.6m$ and $l_2 = 0.5m$. For these link lengths, the masses, centers of mass and moment of inertia of the links are given by $m_1 = 11.3kg$, $m_2 = 9.4kg$, $l_{c1} = 0.3m$, $l_{c2} = 0.25m$, $I_1 = 0.3676kg.m^2$ and $I_2 = 0.2199kg.m^2$.

- i. Evaluate the general inverse kinematic solution you found in part (a) for the given l_1 and l_2 to find q_n .

Solution: Theta values from IK solution: $\theta_{1,n} = 0.73\text{rads}$ and $\theta_{2,n} = -1.65\text{rads}$.

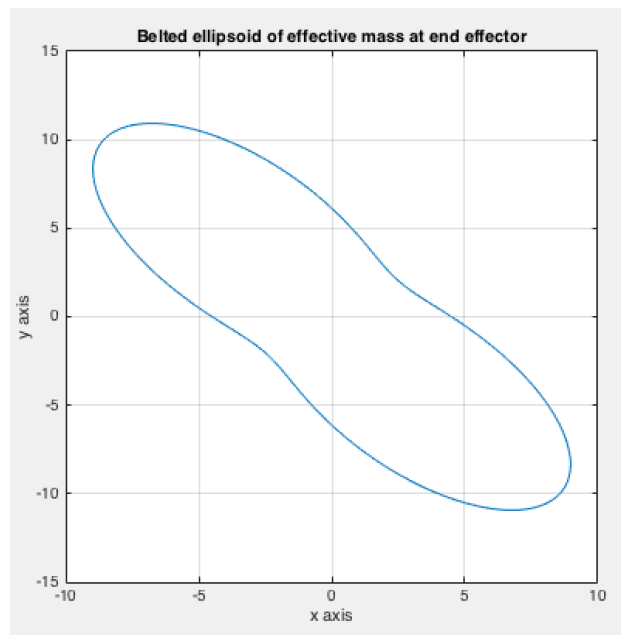
- ii. Draw the belted ellipsoid of effective mass at the end effector in frame $\{0\}$ for the configuration q_n . The linear velocity Jacobian for the end effector in frame $\{0\}$ and the generalized mass matrix are as follows:

$$J_v = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

$$A = \begin{bmatrix} I_1 + I_2 + m_1 l_{c1}^2 + m_2 (l_{c2}^2 + l_1^2 + 2l_1 l_{c2} c_2) & I_2 + m_2 (l_{c2}^2 + l_1 l_{c2} c_2) \\ I_2 + m_2 (l_{c2}^2 + l_1 l_{c2} c_2) & I_2 + m_2 l_{c2}^2 \end{bmatrix}$$

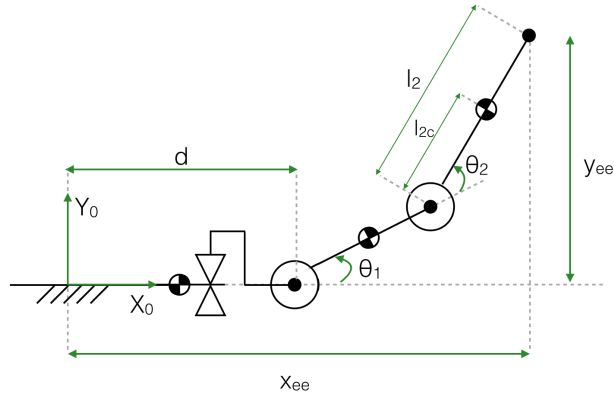
where $s_1 = \sin \theta_1$, $s_{12} = \sin(\theta_1 + \theta_2)$, $c_1 = \cos \theta_1$, $c_{12} = \cos(\theta_1 + \theta_2)$.

Solution: The solution code to plot the belted ellipsoid at the end effector can be found under `cs327a/hw3_sol/p1bc_sol.m`. The resulting effective mass distribution is shown below.



(c) Macro-mini inertial properties

Now, we will explore the effect of mounting the RR manipulator on a moving base, as shown below.



Once again assume that the revolute joints are at the configuration q_n you found above for link lengths equal to the $l_1 = 0.6m$ and $l_2 = 0.5m$.

Assume that the base has a mass $m_0 = 10kg$ and is at position $d = 0m$ for the current analysis.

- i. Find the modified linear velocity Jacobian in frame $\{0\}$ and the modified A matrix.

Solution: The modified linear velocity J_v is obtained simply by taking into account the additional translational freedom along x .

$$J_v = \begin{bmatrix} 1 & -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ 0 & l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$

While it is possible to calculate the A matrix from first principles through the inspecting the total kinetic energy of the three links, three key insights can help to quickly evaluate the modified A matrix:

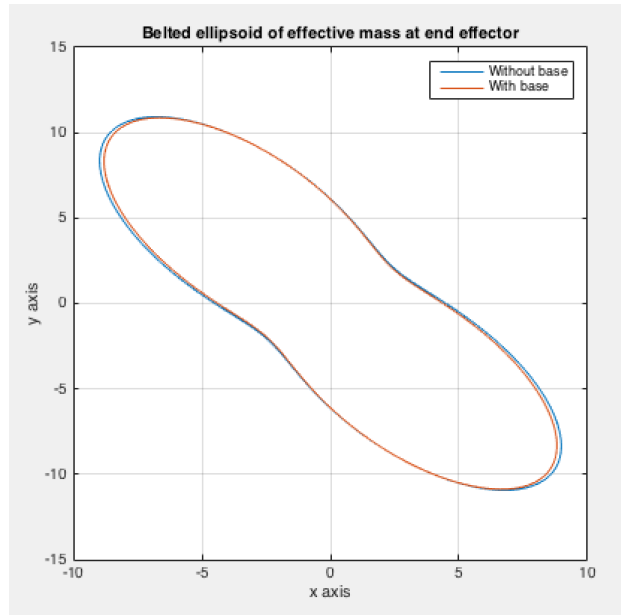
- The bottom-right 2x2 corner of the modified A matrix must be the same as the previous A matrix.
- The top-left entry must simply be the total mass of the system as the additional joint is a prismatic joint.
- Finally, the additional off-diagonal elements can be found from evaluating the inertial reaction forces on the base joint from accelerating one of the two revolute joints and keeping the other one fixed.

In either case, you get the following modified A matrix:

$$A = \begin{bmatrix} m_0 + m_1 + m_2 & -m_1 l_{c1} s_1 - m_2 (l_1 s_1 + l_{c2} s_{12}) & -m_2 l_{c2} s_{12} \\ -m_1 l_{c1} s_1 - m_2 (l_1 s_1 + l_{c2} s_{12}) & I_1 + I_2 + m_1 l_{c1}^2 + m_2 (l_{c2}^2 + l_1^2 + 2l_1 l_{c2} c_2) & I_2 + m_2 (l_{c2}^2 + l_1 l_{c2} c_2) \\ -m_2 l_{c2} s_{12} & I_2 + m_2 (l_{c2}^2 + l_1 l_{c2} c_2) & I_2 + m_2 l_{c2}^2 \end{bmatrix}$$

- ii. Find the belted ellipsoid of effective mass at the end effector, again in frame $\{0\}$, and overlay on top, the belted ellipsoid you found in part (b). Explain your observation.

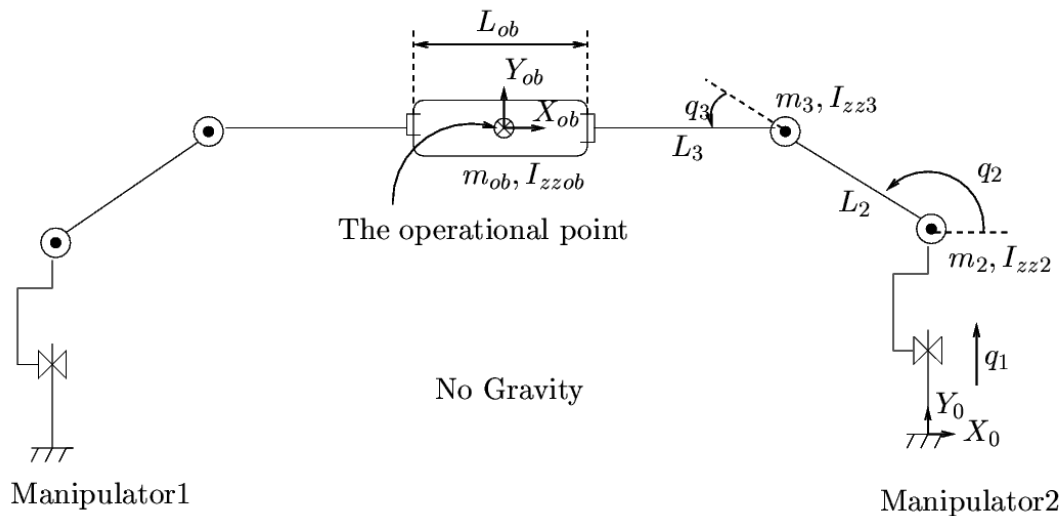
Solution: The resulting effective mass distribution, overlaid on top of the distribution from part b is shown below.



As expected, the new mass distribution is completely contained within that for the RR manipulator alone. Also, the primary reduction is along the x-direction which is justified given that adding the prismatic joint allows additional motion in the x-direction.

Problem 2 - Augmented Object Model/Virtual Linkage

Let's consider two planar PRR-manipulators with an object. The task is to *position* and *orient* the object. Both PRR-manipulators are the same: $m_{object} = 1kg$, $I_{zz,object} = 1kgm^2$ and $L_{object} = 1m$.



The physical properties of each PRR-manipulator are $m_2 = m_3 = 1kg$, $I_{zz2} = I_{zz3} = 1kgm^2$, and $L_2 = L_3 = 1m$ (you may assume the prismatic joint is massless). The corresponding mass matrix and Jacobian for the end-effector are:

$$J = \begin{bmatrix} 0 & -L_2 \sin(q_2) - L_3 \sin(q_2 + q_3) & -L_3 \sin(q_2 + q_3) \\ 1 & L_2 \cos(q_2) + L_3 \cos(q_2 + q_3) & L_3 \cos(q_2 + q_3) \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -s_2 - s_{23} & -s_{23} \\ 1 & c_2 + c_{23} & c_{23} \\ 0 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} m_2 + m_3 & m_3 L_2 c_2 & 0 \\ m_3 L_2 c_2 & m_3 L_2^2 + I_{zz2} + I_{zz3} & I_{zz3} \\ 0 & I_{zz3} & I_{zz3} \end{bmatrix} = \begin{bmatrix} 2 & c_2 & 0 \\ c_2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

The Jacobian for the operational point is

$$J_{ob} = \begin{bmatrix} 0 & -L_2 \sin(q_2) - (L_3 + \frac{L_{ob}}{2}) \sin(q_2 + q_3) & -(L_3 + \frac{L_{ob}}{2}) \sin(q_2 + q_3) \\ 1 & L_2 \cos(q_2) + (L_3 + \frac{L_{ob}}{2}) \cos(q_2 + q_3) & (L_3 + \frac{L_{ob}}{2}) \cos(q_2 + q_3) \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -s_2 - 1.5s_{23} & -1.5s_{23} \\ 1 & c_2 + 1.5c_{23} & 1.5c_{23} \\ 0 & 1 & 1 \end{bmatrix}$$

Let's consider the configuration where $q_1 = 1m$, $q_2 = 30^\circ$, $q_3 = -30^\circ$ for the manipulator1 and $q_1 = 1m$, $q_2 = 150^\circ$, $q_3 = 30^\circ$ for the manipulator2.

- i. Calculate the pseudo kinetic energy matrix Λ_{\oplus} for this augmented object.

Solution:

$$\Lambda_{\oplus} = \Lambda_1 + \Lambda_2 + \Lambda_{ob}.$$

$$\Lambda_1 = (J_{ob1} A^{-1} J_{ob1}^T)^{-1}, \text{ where } q_1 = 1m, q_2 = 30^\circ \text{ and } q_3 = -30^\circ.$$

$$\Lambda_2 = (J_{ob2} A^{-1} J_{ob2}^T)^{-1}, \text{ where } q_1 = 1m, q_2 = 150^\circ \text{ and } q_3 = 30^\circ.$$

$$\Lambda_{ob} = \begin{bmatrix} m_{ob} & 0 & 0 \\ 0 & m_{ob} & 0 \\ 0 & 0 & I_{zzob} \end{bmatrix}$$

$$\Lambda_{\oplus} = \begin{bmatrix} 17.0000 & 0.0000 & -5.1962 \\ 0.0000 & 5.0000 & -0.0000 \\ -5.1962 & -0.0000 & 12.0000 \end{bmatrix}$$

- ii. Find the W matrix, which relates resultant forces/moments and the applied forces/moments at the grasp points in the local frame of the object, i.e.

$$\begin{bmatrix} f_r \\ m_r \end{bmatrix} = \begin{bmatrix} f_{r,x} \\ f_{r,y} \\ m_r \end{bmatrix} = W \begin{bmatrix} f_{1,x} \\ f_{1,y} \\ f_{2,x} \\ f_{2,y} \\ m_1 \\ m_2 \end{bmatrix}, \text{ where } W = [W_f \ W_m].$$

Solution:

$$W_f = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -\frac{L_{ob}}{2} & 0 & \frac{L_{ob}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -0.5 & 0 & 0.5 \end{bmatrix}$$

$$W_m = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$W = [W_f \ W_m] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -0.5 & 0 & 0.5 & 1 & 1 \end{bmatrix}$$

iii. Find E matrix which relates the applied forces at the grasp points with internal forces, i.e.

$$\begin{bmatrix} f_{1,x} \\ f_{1,y} \\ f_{2,x} \\ f_{2,y} \end{bmatrix} = Et, \text{ where } t \text{ is tension between two grasping points.}$$

Solution:

$$E = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\bar{E} = \begin{bmatrix} -0.5 & 0 & 0.5 & 0 \end{bmatrix}$$

iv. Compute the *Grasp Description Matrix*, G .

Solution:

$$\begin{bmatrix} f_{r,x} \\ f_{r,y} \\ m_r \\ t \\ m_1 \\ m_2 \end{bmatrix} = G \begin{bmatrix} f_{1,x} \\ f_{1,y} \\ f_{2,x} \\ f_{2,y} \\ m_1 \\ m_2 \end{bmatrix}, \text{ where } G = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -0.5 & 0 & 0.5 & 1 & 1 \\ -0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$