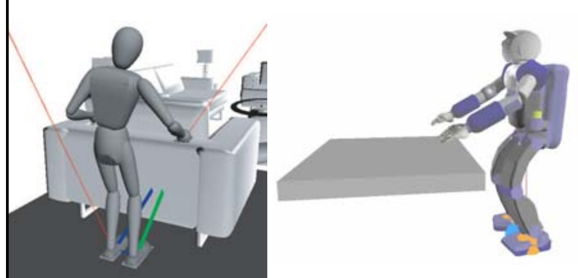
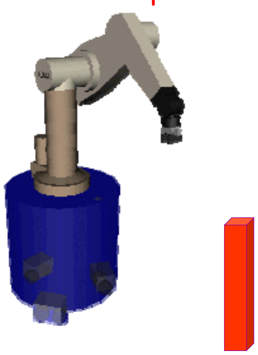
Operational Space Framework!

Whole-body Compliance!

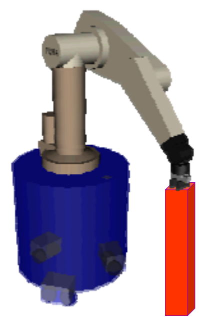


.. motion in contact

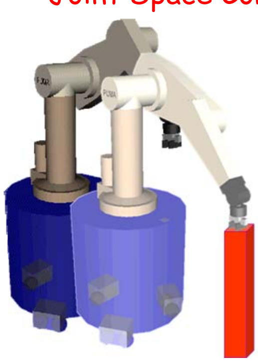
Joint Space Control



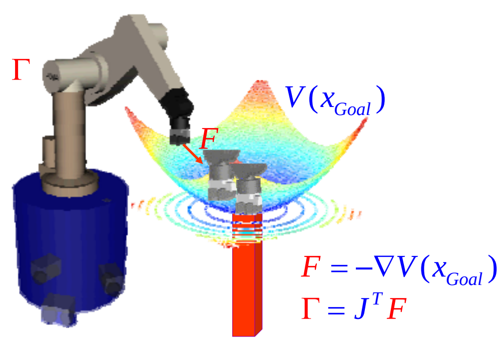
Joint Space Control



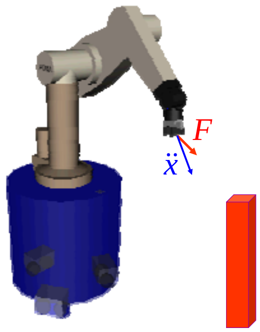
Joint Space Control



Task-Oriented Control



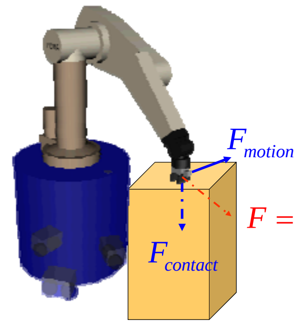
Task-Oriented Dynamics



$$\Lambda\ddot{x} + \mu + p = F$$

$$F = F(\text{dynamics})$$

Unified Motion & Force Control



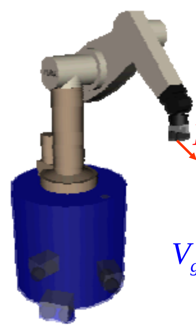
$$F = F_{motion} + F_{contact}$$

Equations of Motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F$$

with $L(x,\dot{x}) = T(x,\dot{x}) - V(x)$

End-Effector Control



$$\Gamma = J^T(q)F$$

$$F = -\nabla V(x_{Goal})$$

$$V_{goal} = \frac{1}{2}k_p\left(x - x_g\right)^T\left(x - x_g\right)$$

Passive Systems

$$U_{goal} = \frac{1}{2}k_p\left(x - x_g\right)^T\left(x - x_g\right)$$

System $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial\left(T - V\right)}{\partial x} = F$

$$\Downarrow \quad F = -\frac{\partial}{\partial X}\left(V_{goal} - \hat{V}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial\left(T - V_{goal}\right)}{\partial x} = 0$$

Stable

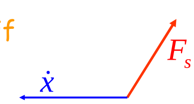
Conservative Forces

Asymptotic Stability

a system $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial\left(T - V_{goal}\right)}{\partial x} = F_s$

is asymptotically stable if

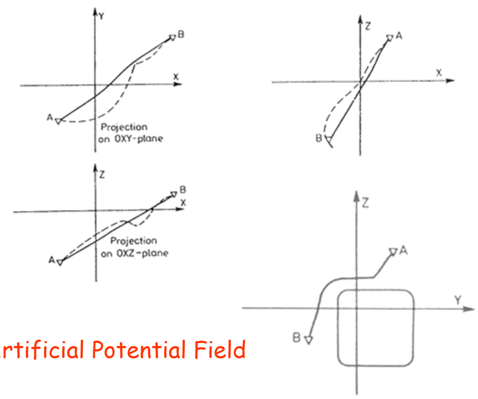
$F_s^T\dot{x} < 0 \;\; ; \; for \; \dot{x} \neq 0$



$F_s = -k_v\dot{x} \;\; \rightarrow \;\; k_v > 0$

Control

$$F = -k_p\left(x - x_g\right) - k_v\dot{x} + \hat{p}$$



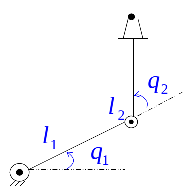
Artificial Potential Field

Operational Space Dynamics

$$\Lambda(x)\ddot{x} + \mu(x,\dot{x}) + p(x) = F$$

- x: End-Effector Position and Orientation
- $\Lambda(x)$: End-Effector Kinetic Energy Matrix
- $\mu(x,\dot{x})$: End-Effector Centrifugal and Coriolis forces
- $p(x)$: End-Effector Gravity forces
- F: End-Effector Generalized forces

Example 2-d.o.f arm



$$\Lambda(x)\ddot{x} + \mu(x,\dot{x}) + p(x) = F$$

$$F = -k_p\left(x - x_g\right) - k_v\,\dot{x} + \hat{p}(x)$$

$$\left(m_1^* c^2 12 + m_2\right)\ddot{x} + m_1^*\ddot{y} + \mu_1 = -k_p\left(x - x_g\right) - k_v\dot{x}$$

$$\left(m_1^* c^2 12 + m_2\right)\ddot{y} + m_1^*\ddot{x} + \mu_2 = -k_p\left(y - y_g\right) - k_v\dot{y}$$

Closed loop behavior

$$m_{11}(q)\ddot{x} + k_v\dot{x} + k_p\left(x - x_g\right) = -\left(m_1^*\ddot{y} + \mu_1\right)$$

$$m_{22}(q)\ddot{y} + k_v\dot{y} + k_p\left(y - y_g\right) = -\left(m_1^*\ddot{x} + \mu_2\right)$$

Joint Space Dynamics

$$A(q)\ddot{q} + b(q,\dot{q}) + g(q) = \Gamma$$

- q: Joint Coordinates
- $A(q)$: Kinetic Energy Matrix
- $b(q,\dot{q})$: Centrifugal and Coriolis forces
- $g(q)$: Gravity forces
- Γ: Generalized forces

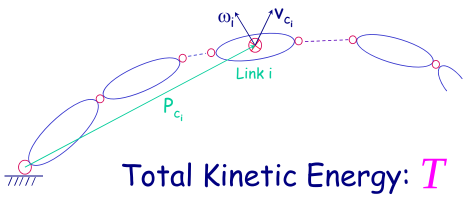
Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

$$A(q)\ddot{q} + b(q,\dot{q}) + g(q) = \Gamma$$

$$A(q): T = \frac{1}{2}\dot{q}^T A\,\dot{q} \qquad A(q) \Rightarrow b(q,\dot{q})$$

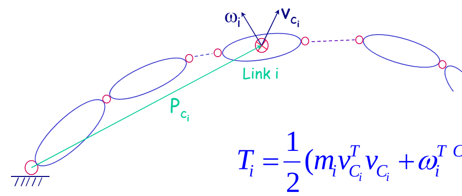
Equations of Motion



Total Kinetic Energy: T

$$T = \sum T_{Link\,i} \equiv \frac{1}{2}\dot{q}^T A\, \dot{q}$$

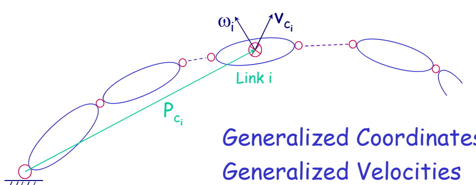
Equations of Motion — Explicit Form



$$T_i = \frac{1}{2}(m_i v_{C_i}^T v_{C_i} + \omega_i^T\, {}^C I_i \omega_i)$$

Total Kinetic Energy ⇨ $T = \sum_{i=1}^{n} T_i$

Equations of Motion — Explicit Form



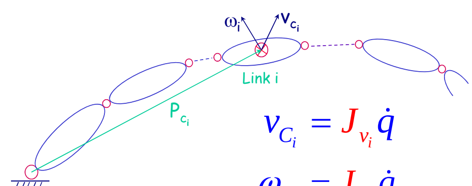
Generalized Coordinates q

Generalized Velocities $\dot{q}$

Kinetic Energy Quadratic Form of Generalized Velocities $T = \frac{1}{2}\dot{q}^T A\, \dot{q}$

$$\frac{1}{2}\dot{q}^T A\, \dot{q} \equiv \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T\, {}^C I_i \omega_i)$$

Equations of Motion — Explicit Form



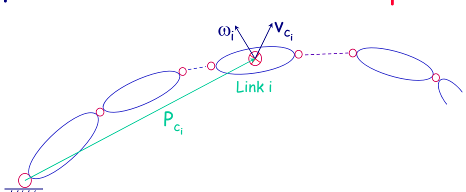
$$v_{C_i} = J_{v_i}\dot{q}$$

$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

$$\frac{1}{2}\dot{q}^T A\, \dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T\, {}^C I_i \omega_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n}(m_i \dot{q}^T J_{v_i}^T J_{v_i}\dot{q} + \dot{q}^T J_{\omega_i}^T\, {}^C I_i J_{\omega_i}\dot{q})$$

Equations of Motion — Explicit Form



$$\frac{1}{2}\dot{q}^T A\, \dot{q} = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T\, {}^C I_i J_{\omega_i})\right]\dot{q}$$

$$A = \sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T\, {}^C I_i J_{\omega_i})$$

$$\underset{(n\times n)}{A(q)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Christoffel Symbols

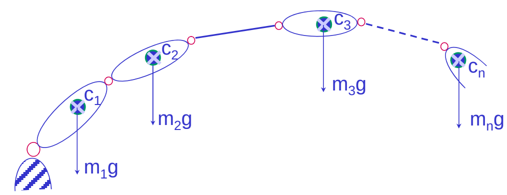
$$b_{ijk} = \frac{1}{2}(a_{ijk} + a_{ikj} - a_{jki}) \qquad a_{ijk} = \frac{\partial a_{ij}}{\partial q_k}$$

$$b(q,\dot{q}) = C(q)\left[\dot{q}^2\right] + B(q)\left[\dot{q}\dot{q}\right]$$

$$\underset{(n\times n)}{C(\mathbf{q})}\underset{(n\times 1)}{[\dot{\mathbf{q}}^2]} = \begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

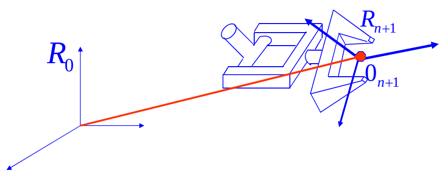
$$\underset{(n\times\frac{(n-1)n}{2})}{B(\mathbf{q})}\ \underset{(\frac{(n-1)n}{2}\times 1)}{[\dot{\mathbf{q}}\dot{\mathbf{q}}]} = \begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \vdots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix} \begin{bmatrix} \dot{q}_1\dot{q}_2 \\ \dot{q}_1\dot{q}_3 \\ \vdots \\ \dot{q}_{(n-1)}\dot{q}_n \end{bmatrix}$$

Gravity Vector



$$g = -(J_{v_1}^T(m_1 g) + J_{v_2}^T(m_2 g) + \cdots + J_{v_n}^T(m_n g))$$

Effector Equations of Motion



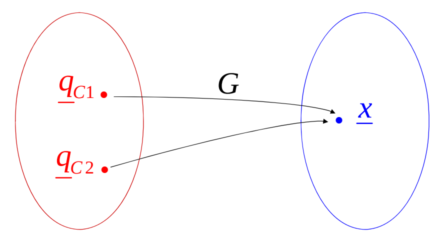
Non-Redundant Manipulator ; $n = m_0$

$$x = \left(x_1\, x_2 \ldots x_{m_0}\right)^T$$

$$q = \left(q_1\, q_2 \ldots q_n\right)^T$$

$$x = G\left(q\right)$$

Domain



$$D_q = \prod_{i=1}^{n}\left[\underline{q}_i, \bar{q}_i\right] \qquad D_x = G\left(D_q\right)$$

$$\tilde{D}_x = G\left(\tilde{D}_q\right)$$

$\tilde{D}_q$: Excluding Singularities and such that G is one-to-one

In $\tilde{D}_x$, x_1, x_2, … x_{m_0} form a complete set of configuration parameters for the manipulator.

$x_1, \ldots, x_{m_0}$: system of generalized coordinates

Kinetic Energy

$$T\left(x, \dot{x}\right) = \frac{1}{2}\dot{x}^T \Lambda(x)\,\dot{x}$$

$\Lambda_{m_0\times m_0}(x)$: Kinetic Energy Matrix

$$T_x(x,\dot{x}) = \frac{1}{2}\dot{x}^T \Lambda(x)\,\dot{x}$$

Identity

$$T_x(x,\dot{x}) \equiv T_q(q,\dot{q})$$

$$\frac{1}{2}\dot{x}^T \Lambda(X)\dot{x} \equiv \frac{1}{2}\dot{q}^T A(q)\,\dot{q}$$

$$\dot{x} = J\dot{q}$$

$$\Lambda(x) = J^{-T}(q)\; A(q)\; J^{-1}(q)$$

$$p(x) = J^{-T} g(q)$$

System $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial (T-U)}{\partial x} = F$

$$\frac{\partial T}{\partial \dot{x}} = \frac{1}{2}\frac{\partial}{\partial \dot{x}}\left(\dot{x}^T \Lambda\, \dot{x}\right) = \Lambda\, \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \dot{\Lambda}\dot{x} + \Lambda\ddot{x}$$

$$\frac{\partial T}{\partial x_i} = \frac{1}{2}\dot{x}^T \Lambda_{x_i}\dot{x}$$

$$\mu(x,\dot{x}) = \dot{\Lambda}\dot{x} - \begin{vmatrix} \frac{1}{2}\dot{x}^T \Lambda_{x_1}\dot{x} \\ \vdots \\ \frac{1}{2}\dot{x}\Lambda_{x_{m0}}\dot{x} \end{vmatrix}$$

$$\mu(x,\dot{x}) = \dot{\Lambda}\dot{x} - m(x,\dot{x})$$

$$m_i = \frac{1}{2}\dot{x}^T \Lambda_{x_i}\dot{x} \qquad ; \quad \Lambda = J^{-T} A J^{-1}$$

$$\begin{cases} \dot{\Lambda}\dot{x} = J^{-T}\dot{A}\dot{q} - \Lambda h(q,\dot{q}) + \dot{J}^{-T} A\dot{q} \\ m(x,\dot{x}) = J^{-T} l(q,\dot{q}) + \dot{J}^{-T} A\dot{q} \end{cases}$$

where $h \triangleq \dot{J}\dot{q}$

$$l_i \triangleq \frac{1}{2}\dot{q}^T A_{q_i}\dot{q}$$

$$m_i(x,\dot{x}) = \frac{1}{2}\dot{x}^T \Lambda_{x_i}\dot{x}$$

$$m(x,\dot{x}) = \frac{1}{2}|\,\dot{x}^T J^{-T} A_{x_i} J^{-1}\dot{x}\,| + 2\,\frac{1}{2}|\,\dot{x}^T J_{x_i}^{-T} A J^{-1}\dot{x}\,|$$

$$m(x,\dot{x}) = \frac{1}{2}|\,\dot{q}^T A_{x_i}\dot{q}\,| + \dot{J}^{-T} A\,\dot{q}$$

$$\dot{q}^T A_{x_i}\dot{q} = \left(\frac{\partial q_1}{\partial x_i}\frac{\partial q_2}{\partial x_i}\cdots\frac{\partial q_n}{\partial x_i}\right).\,|\,\dot{q}^T A_{q_i}\dot{q}\,|$$

$$\frac{1}{2}|\,\dot{q}^T A_{x_i}\dot{q}\,| = J^{-T} l(q,\dot{q})$$

$$m(x,\dot{x}) = J^{-T} l(q,\dot{q}) + \dot{J}^{-T} A\,\dot{q}$$

$$\mu = J^{-T}\left(\dot{A}\dot{q} - l(q,\dot{q})\right) - \Lambda h(q,\dot{q})$$

where $h \triangleq \dot{J}\dot{q}$

$$l_i \triangleq \frac{1}{2}\dot{q}^T A_{q_i}\dot{q}$$

$$\mu = J^{-T}(q)\; b(q,\dot{q}) - \Lambda h(q,\dot{q})$$

Joint Space/Operational Space Relationships

$$T_x(x,\dot{x}) \equiv T_q(q,\dot{q})$$

$$\frac{1}{2}\dot{x}^T \Lambda(X)\dot{x} = \frac{1}{2}\dot{q}^T A(q)\dot{q}$$

Using $\dot{x} = J(q)\dot{q}$

$$\frac{1}{2}\dot{q}^T J^T \Lambda\, J\, \dot{q} = \frac{1}{2}\dot{q}^T A\, \dot{q}$$

Joint Space/Operational Space Relationships

$$\Lambda(x) = J^{-T}(q)\, A(q)\, J^{-1}(q)$$

$$\mu(x,\dot{x}) = J^{-T}(q)\, b(q,\dot{q}) - \Lambda(q)\, h(q,\dot{q})$$

$$p(x) = J^{-T}(q)\, g(q)$$

where $h(q,\dot{q}) \doteq \dot{J}(q)\dot{q}$

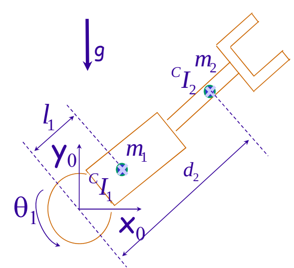
Λ, μ, and P are all expressed in terms of joint coordinates

The domain $\tilde{D}_x$ can be extended to

$$\bar{D}_x = G\left(\bar{D}_q\right)$$

$\bar{D}_q$: domain D_q excluding singularities

Example



$$q_2 = d_2$$

$$x = \begin{bmatrix} d_2 c1 \\ d_2 s1 \end{bmatrix}$$

$$^0J = \begin{bmatrix} -d_2 s1 & c1 \\ d_2 c1 & s1 \end{bmatrix}$$

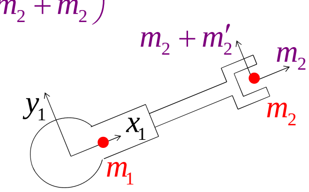
$$^0J = \begin{bmatrix} -d_2 s1 & c1 \\ d_2 c1 & s1 \end{bmatrix}$$

$$^0J = \begin{pmatrix} c1 & -s1 \\ s1 & c1 \end{pmatrix} \overbrace{\begin{pmatrix} 0 & 1 \\ d_2 & 0 \end{pmatrix}}^{^1J}$$

$$^1J^{-1} = \begin{pmatrix} 0 & 1/d_2 \\ 1 & 0 \end{pmatrix};$$

$$^1\Lambda = \begin{pmatrix} 0 & 1 \\ 1/d_2 & 0 \end{pmatrix} \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix} \begin{pmatrix} 0 & 1/d_2 \\ 1 & 0 \end{pmatrix}$$

$$^1\Lambda = \begin{pmatrix} m_2 & 0 \\ 0 & m_2 + m_2' \end{pmatrix}$$

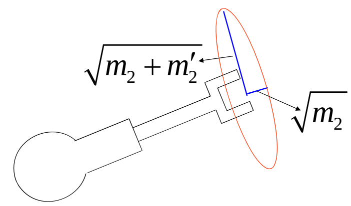


$$m_2' = \frac{I_{221} + I_{222} + m_1 l_1^2}{d_2^2}$$

$$ {}^0\Lambda = \begin{pmatrix} c1 & -s1 \\ s1 & c1 \end{pmatrix} \begin{pmatrix} m_2 & 0 \\ 0 & m_2^+ \end{pmatrix} \begin{pmatrix} c1 & s1 \\ -s1 & c1 \end{pmatrix} $$

$$ m_2^+ = m_2 + m_2' $$

$$ {}^0\Lambda = \begin{pmatrix} m_2 + m_2' s1^2 & -m_2' s\,c1 \\ -m_2' s\,c1 & m_2 + m_2' c1^2 \end{pmatrix} $$



$$ {}^0\Lambda = \begin{pmatrix} m_2 + m_2' s1^2 & -m_2' s\,c1 \\ -m_2' s\,c1 & m_2 + m_2' c1^2 \end{pmatrix} $$

Nonlinear Dynamic Decoupling

Model

$$ \Lambda(x)\ddot{x} + \mu(x,\dot{x}) + p(x) = F $$

Control Structure

$$ F = \hat{\Lambda}(x)F^* + \hat{\mu}(x,\dot{x}) + \hat{p}(x) $$

Decoupled System

$$ I\,\ddot{x} = F^* $$

with $\Gamma = J^T F$

Dynamic Decoupling

$$ \Lambda(x)\ddot{x} + \mu(x,\dot{x}) + p(x) = F $$

$$ F = \hat{\Lambda}F^* + \hat{\mu}(x,\dot{x}) + \hat{p}(x) $$

$$ I_{m_0}\ddot{X} = \underbrace{\left(\Lambda^{-1}\hat{\Lambda}\right)}_{G(x)} F^* + \underbrace{\Lambda^{-1}\left(\hat{\mu} - \mu\right)}_{\tilde{\mu}_{(x,\dot{x})}} + \underbrace{\Lambda^{-1}\left(P - \hat{P}\right)}_{\tilde{P}(x)} $$

$$ I_{m_0}\ddot{x} = G(x)F^* + \varepsilon(x,\dot{x}) + d(t) $$

$$ G(x) = \Lambda^{-1}\hat{\Lambda} \simeq I + \varepsilon_\Lambda $$

$$ \varepsilon(x,\dot{x}) = \Lambda^{-1}\left(\tilde{\mu} + \tilde{P}\right) $$

$d(t)$: unmodeled disturbances

Perfect Estimates

$$ I_{m_0}\ddot{x} = F^* $$

F^* input of decoupled end-effector

Goal Position Control

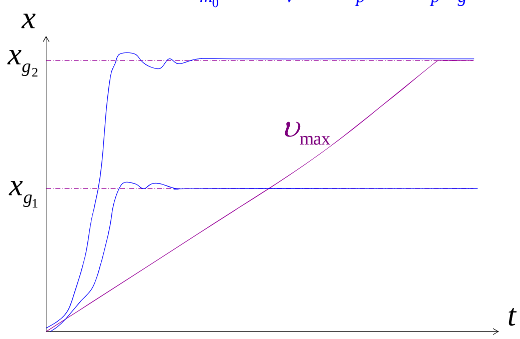
$$ F^* = -k_v\dot{x} - k_p\left(x - x_g\right) $$

Closed Loop

$$ I_{m_0}\ddot{x} + k_v\dot{x} + k_p x = k_p x_g $$

Closed Loop

$$I_{m_0}\,\ddot{x} + k_v \dot{x} + k_p x = k_p x_g$$

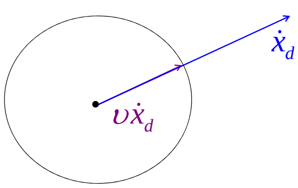


PD Control

$$F^* = -k_v \dot{x} - k_p \left(x - x_g\right)$$

Velocity-Like Control

$$F^* = -k_v \left(\dot{x} - \frac{k_p}{k_v}\left(x_g - x\right) \right)$$



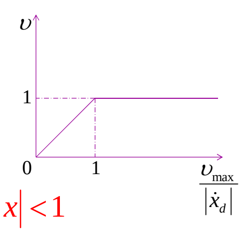
$$\dot{x}_d = \frac{k_p}{k_v}\left(x_g - x\right)$$

$$F^* = -k_v \left(\dot{x} - \underbrace{\frac{k_p}{k_v}\left(x_g - x\right)}_{\dot{x}_d} \right)$$

$$F^* = -k_v \left(\dot{x} - \upsilon \dot{x}_d\right)$$

with

$$\upsilon = sat\left(\frac{V_{\max}}{|\dot{x}_d|}\right)$$



$$sat(x) = \begin{cases} x & if\ |x| < 1 \\ sign(x) & if\ |x| > 1 \end{cases}$$

Trajectory Tracking

Trajectory: x_d, $\dot{x}_d$, $\ddot{x}_d$

$$F^* = I_{m_0}\ddot{x}_d - k_v(\dot{x} - \dot{x}_d) - k_p(x - x_d)$$

$$(\ddot{x} - \ddot{x}_d) + k_v(\dot{x} - \dot{x}_d) + k_p(x - x_d)$$

or $\boxed{\ddot{\varepsilon}_X + k_v \dot{\varepsilon}_X + k_p \varepsilon_X = 0}$

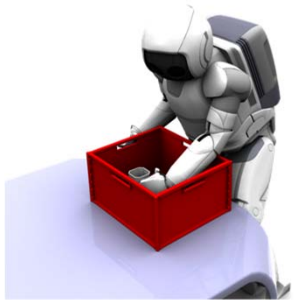
with $\varepsilon_X = x - x_d$

In joint space

$$\ddot{\varepsilon}_q + k_v \dot{\varepsilon}_q + k_p \varepsilon_q = 0$$

with $\varepsilon_q = q - q_d$

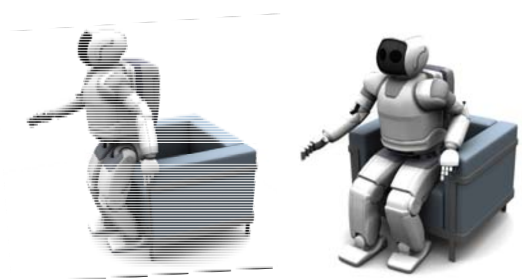
Compliant Motion Control

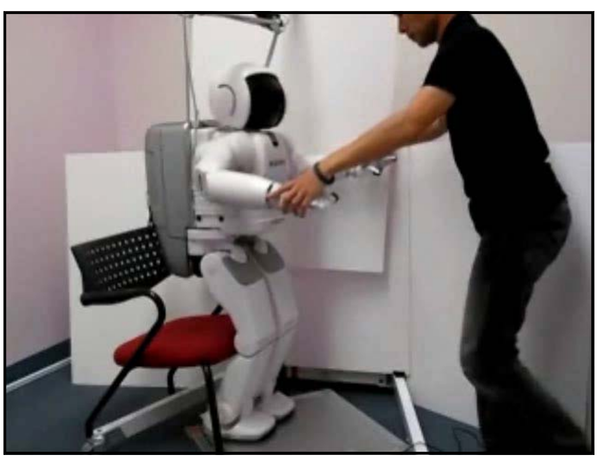


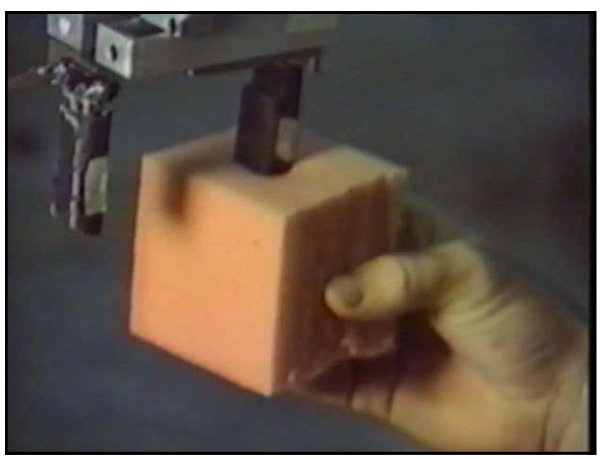
Advanced Manipulation Capabilities

Compliant Manipulation Primitives

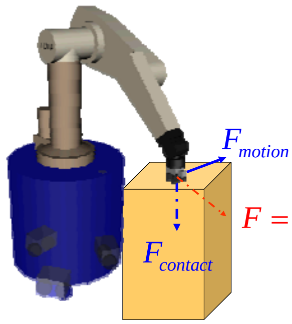
Multi-contact Manipulation





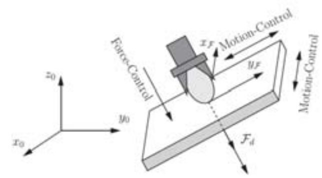


Unified Motion/Force Control



$F = F_{motion} + F_{contact}$

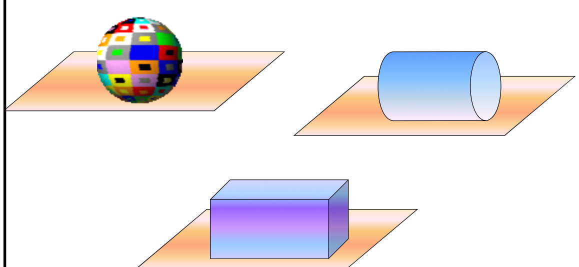
Unified Motion/Force Control



- Generalized Selection Matrix
- Dynamic Model (Homogeneity)

$$\Lambda_0(x)\dot{\vartheta} + \mu_o(x,\vartheta) + p_0(x) + F_{contact} = F_0$$

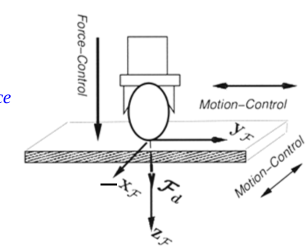
Task Description



Task Specification

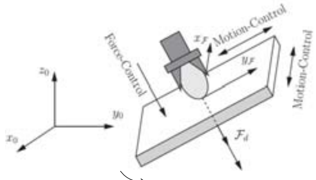
$$F = \Sigma F_{motion} + \bar{\Sigma} F_{force}$$

Selection matrix



$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \ \bar{\Sigma} = I - \Sigma$$

Generalized Selection Matrix



$f_{(R_f)} = R_f f_{(R_0)}$

Selection in R_f

$R_f \quad \Sigma \rightarrow \Omega$

$\bar{\Sigma} f_{R_f}$

Selection in R_0

$$\underbrace{R_f^T \bar{\Sigma} R_f}_{\bar{\Omega}} f_{R_0} \Leftarrow R_f^T \bar{\Sigma} f_{R_f}$$

Generalized Selection Matrix

$$\Sigma_F = \begin{pmatrix} \sigma_{F_x} & 0 & 0 \\ 0 & \sigma_{F_y} & 0 \\ 0 & 0 & \sigma_{F_z} \end{pmatrix}; \quad \bar{\Sigma}_F = I_3 - \Sigma_F$$

$$\Sigma_M = \begin{pmatrix} \sigma_{M_x} & 0 & 0 \\ 0 & \sigma_{M_y} & 0 \\ 0 & 0 & \sigma_{M_z} \end{pmatrix}; \quad \bar{\Sigma}_M = I_3 - \Sigma_M$$

Generalized Selection Matrix

$$\Omega = \begin{pmatrix} R_F^T \; \Sigma_F \; R_F & 0 \\ 0 & R_M^T \; \Sigma_M \; R_M \end{pmatrix}$$

$$\bar{\Omega} = \begin{pmatrix} R_F^T \; \bar{\Sigma}_F \; R_F & 0 \\ 0 & R_M^T \; \bar{\Sigma}_M \; R_M \end{pmatrix}$$

Basic Dynamic Model

Operational force $\overset{?}{\equiv}$ Forces & Moments

$$\dot{x} = J(q)\dot{q}$$

$$\Gamma = J^T(q)F$$

Linear & Angular Velocities | Forces & Moments

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q)\dot{q}$$

$$\vartheta \triangleq \begin{pmatrix} v \\ \omega \end{pmatrix} \qquad F_0 \triangleq \begin{pmatrix} f \\ m \end{pmatrix}$$

$$\dot{x} = J\,\dot{q} = E(x)J_0\,\dot{q} \qquad \tau = J^T F = J_0^T(E^T F)$$

$$\dot{x} = E\begin{pmatrix} v \\ \omega \end{pmatrix} \qquad \begin{pmatrix} f \\ m \end{pmatrix} = E^T\,F$$

Basic Dynamic Model

$$\Lambda_0 = E^T \Lambda\, E$$

$$\Lambda\,\ddot{x} + \mu(x,\dot{x}) + p(x) = F$$

$$\Downarrow E^T$$

$$\boxed{\Lambda_0\,\dot{\vartheta} + \mu_0(x,\vartheta) + p_0(x) = F_0}$$

with $\vartheta \triangleq \begin{pmatrix} v \\ \omega \end{pmatrix}$

Orientation Representation

$$x_r,\; x_{rd} \rightarrow \delta x_r = x_r - x_{rd}$$

$$\dot{x}_r = E_r\,\omega$$

Instantaneous Angular Error

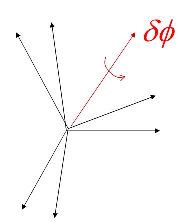
$$\delta x_r = E_r\,\delta\phi$$

Instantaneous Angular Error

$$\dot{x}_r = E_r\omega$$

$$\delta x_r = E_r\,\delta\phi$$

$$\boxed{\delta\phi = E_r^+\,\delta x_r}$$



Control - Position Errors

$$(x - x_d) = \begin{pmatrix} x_p - x_{pd} \\ x_r - x_{rd} \end{pmatrix}$$

$$\dot{x}_r = E_r\,\omega$$

$$\underline{\delta x_r} = E_r\,\underline{\delta\phi} \;\Rightarrow\; \begin{pmatrix} x_p - x_{pd} \\ \delta\phi \end{pmatrix} \text{ Error Vector}$$

$$\delta x_r = \left(x_r - x_{r(d)}\right) = E_r\,\delta\phi$$

$$\boxed{\delta\phi = E_r^+\left(x_r - x_{r(d)}\right)}$$

Goal Position

$$x_d = \begin{bmatrix} x_{pd} \\ x_{rd} \end{bmatrix}$$

$$f^* = -k_p\left(x_p - x_{pd}\right) - k_v\dot{x}_p$$

$$m^* = -k_p\delta\phi - k_v\omega$$

with $\delta\phi = E_r^+(x_r)(x_r - x_{rd})$

Closed loop

$$I.\ddot{x}_p + k_v\dot{x}_p + k_p\left(x_p - x_{pd}\right) = 0$$

$$I\,\dot{\omega} + k_v\omega + k_p\delta\phi = 0$$

Direction Cosines

$$x_r = \left(r_1^T r_2^T r_3^T\right)^T$$

$$x_{rd} = \left(r_{1d}^T\, r_{2d}^T\, r_{3d}^T\right)^T$$

$$E_r^+ = \frac{1}{2}E_r^T$$

The angular rotation error

$$\delta\phi = -\frac{1}{2}\left(\hat{r}_1 r_{1d} + \hat{r}_2 r_{2d} + \hat{r}_3 r_{3d}\right)$$

Euler Parameters

The end-effector orientation

$$x_r = \lambda = \left(\lambda_0\lambda_1\lambda_2\lambda_3\right)^T$$

The desired orientation

$$\lambda_d = \left(\lambda_{0d}\lambda_{1d}\lambda_{2d}\lambda_{3d}\right)^T$$

The angular rotation error

$$\boxed{\delta\phi = E_R^+(\lambda)\lambda_d} \qquad E_R^+(\lambda) = 2\begin{pmatrix} -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}$$

Motion Tracking $(x_{pd}\,,\,\dot{x}_{pd}\,,\,\ddot{x}_{pd})$

$$F^* = \ddot{x}_{pd} - k_p\left(x_p - x_{pd}\right) - k_v\left(\dot{x}_p - \dot{x}_{pd}\right)$$

Closed loop

$$I.\ddot{\varepsilon}_x + k_v\dot{\varepsilon}_x + k_p\varepsilon_x = 0$$

with

$$\varepsilon_{x_p} = x_p - x_{pd}$$

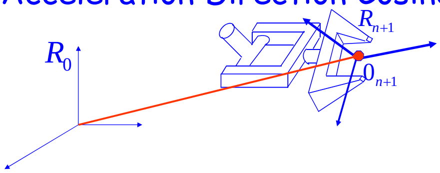
Angular Acceleration

$$\dot{x}_r = E_r\omega$$

$$\ddot{x}_r = E_r\dot{\omega} + \dot{E}_r\omega$$

$$\boxed{\dot{\omega} = E_r^+\ddot{x}_r - E_r^+\dot{E}_r\omega}$$

Acceleration Direction Cosines



The orientation is described by

$$x_r = \left(r_1^T r_2^T r_3^T\right)^T$$

$$r_1 = x_{(n+1)};\ r_2 = y_{(n+1)};\ r_3 = z_{(n+1)};$$

The second time derivatives

$$\frac{d^2\mathbf{x}_{(n+1)}}{dt^2} = -\mathbf{x}_{(n+1)} \times \dot{\omega} + \left(\mathbf{x}_{(n+1)} \times \omega\right) \times \omega$$

$$\frac{d^2\mathbf{y}_{(n+1)}}{dt^2} = -\mathbf{y}_{(n+1)} \times \dot{\omega} + \left(\mathbf{y}_{(n+1)} \times \omega\right) \times \omega$$

$$\frac{d^2\mathbf{z}_{(n+1)}}{dt^2} = -\mathbf{z}_{(n+1)} \times \dot{\omega} + \left(\mathbf{z}_{(n+1)} \times \omega\right) \times \omega$$

However

$$\mathbf{u} \times \mathbf{v} \times \mathbf{w} = \left(\mathbf{u}^T \mathbf{v}\right)\mathbf{w} - \left(\mathbf{v}^T \mathbf{w}\right)\mathbf{u}$$

This yields

$$\ddot{\mathbf{x}}_r = E(\mathbf{x}_r)\dot{\omega} + R(\mathbf{x}_r, \omega)\omega - (\omega^T \omega)\mathbf{x}_r$$

where

$$R(\mathbf{x}_r, \omega) = \begin{pmatrix} \left(r_1^T \omega\right) I_3 \\ \left(r_2^T \omega\right) I_3 \\ \left(r_3^T \omega\right) I_3 \end{pmatrix}$$

Acceleration Direction Cosines

$$\dot{\omega}_d = \frac{1}{2} E^T \ddot{x}_{rd} + \frac{1}{2} R^T (x_r, \omega) \dot{x}_{rd}$$

Euler Parameters

The acceleration associated with Euler parameters

$$\ddot{\lambda} = \frac{1}{4} \check{\lambda} \dot{\omega} - \frac{1}{2} (\omega^T \omega) \lambda$$

since

$$\check{\lambda}^T \lambda = 0 \qquad \check{\lambda} = \begin{pmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix}$$

Euler Parameters

The angular acceleration vector

$$\dot{\omega} = 4 \check{\lambda}^T \ddot{\lambda}$$

The desired angular acceleration

$$\dot{\omega}_d = 4 \check{\lambda}_d^T \ddot{\lambda}_d$$

Motion Tracking $(x_{pd}\ ,\ \dot{x}_{pd}\ ,\ \ddot{x}_{pd})$

$$F^* = \ddot{x}_{pd} - k_p\left(x_p - x_{pd}\right) - k_v\left(\dot{x}_p - \dot{x}_{pd}\right)$$

Closed loop

$$I.\ddot{\varepsilon}_x + k_v\dot{\varepsilon}_x + k_p\varepsilon_x = 0$$

with

$$\varepsilon_{x_p} = x_p - x_{pd}$$

Motion Tracking $(x_{pd}\ ,\ \dot{x}_{pd}\ ,\ \ddot{x}_{pd})$

$$f^* = \ddot{x}_{pd} - k_p\left(x_p - x_{pd}\right) - k_v\left(\dot{x}_p - \dot{x}_{pd}\right)$$

$$m^* = \dot{\omega}_d - k_p\delta\phi - k_v\left(\omega - \omega_d\right)$$

with $\delta\phi = E_r^+\left(x_r - x_{rd}\right)$

$$\omega_d = E_r^+\left(x_{rd}\right)\dot{x}_{rd}$$

and

$$\dot{\omega}_d = E_r^+\left(x_{rd}\right)\ddot{x}_{rd} - E_r^+\left(x_{rd}\right)\dot{E}\left(x_{rd}\right)\omega_d$$

Closed loop

$$\left(\ddot{x}_p - \ddot{x}_{pd}\right) + k_v\left(\dot{x}_p - \dot{x}_{pd}\right) + k_p\left(x_p - x_{pd}\right) = 0$$

$$\left(\dot{\omega} - \dot{\omega}_d\right) + k_v\left(\omega - \omega_d\right) + k_p\delta\phi = 0$$

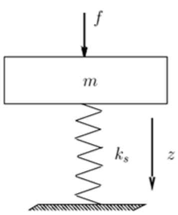
A Mass Spring System

System

$$m\ddot{z} + k_s z = f$$

$$f_s = k_s z$$

$$m\frac{1}{k_s}\ddot{f}_s + f_s = f$$



System $\quad m\frac{1}{k_s}\ddot{f}_s + f_s = f$

Control

$$f = f_s + m \bullet f_{comp}$$

$$f = f_s - m\left[k_f\left(f_s - f_d\right) + k_{v_f}\dot{f}_s\right]$$

Control-loop System

$$\ddot{f}_s + k_s k_{v_f}\dot{f}_s + k_s k_f\left(f_s - f_d\right) = 0$$

Static Equilibrium

$$f_s = f_d$$

End-Effector/Sensor System

$$\Lambda_0\dot{\vartheta} + \mu_0(x,\vartheta) + p_0(x) + F_{contact} = F_0$$

Unified Control

$$F_0 = F_{motion} + F_{force}$$

$$F_{motion} = \hat{\Lambda}_0\,\Omega\,F^*_{motion} + \hat{\mu}_0 + \hat{p}_0$$

$$F_{force} = \hat{\Lambda}_0\,\bar{\Omega}\,F^*_{force} + F_{sensor}$$

End-Effector/Sensor System

$$\Lambda_0 \dot{\vartheta} + \mu_0(x, \vartheta) + p_0(x) + F_{contact} = F_0$$

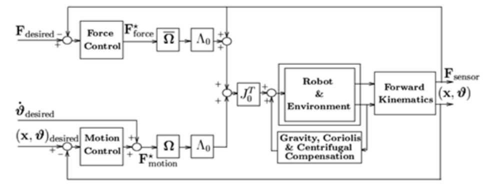
Unified Control

$$F_0 = F_{motion} + F_{force}$$

$$F_{motion} = \hat{\Lambda}_0 \, \Omega \, F^*_{motion} + \hat{\mu}_0 + \hat{P}_0$$

$$F_{force} = \hat{\Lambda}_0 \, \bar{\Omega} \, F^*_{force} + \bar{\Omega} \, F_{desired}$$

Unified Motion & Force Control

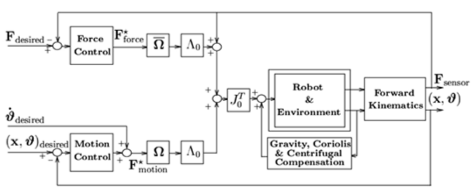


Two decoupled Subsystems

$$\Omega \, \dot{\vartheta} = \Omega \, F^*_{motion}$$

$$\bar{\Omega} \, \dot{\vartheta} = \bar{\Omega} F^*_{force}$$

Unified Motion & Force Control



Two decoupled Subsystems

$$\Omega \, \dot{\vartheta} = \Omega \, F^*_{motion}$$

$$\bar{\Omega} \, \dot{\vartheta} = \bar{\Omega} F^*_{force}$$