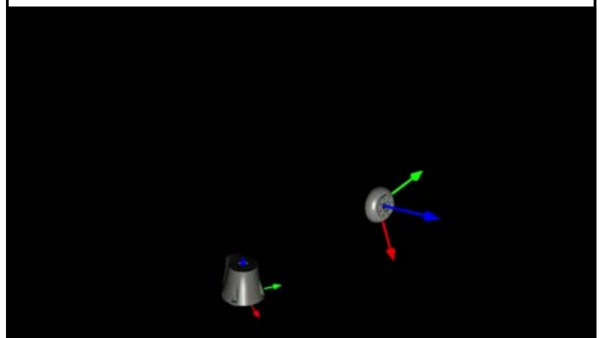
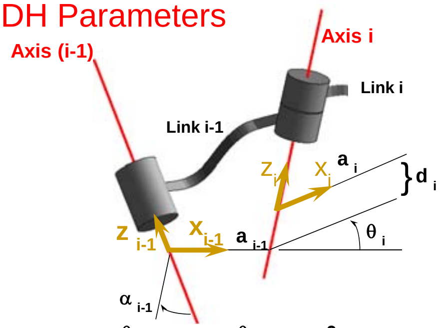
Kinematics, Dynamics & Control



DH Parameters



$$
{}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Velocity/Force Duality

$$\dot{x} = J\ \dot{\theta}$$

$$\tau = J^T F$$

Representations

$$x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$$

x_P:
- •Cartesian
- •Spherical
- •Cylindrical
- •....

x_R:
- •Euler Angles
- •Direction Cosines
- •Euler Parameters

Jacobian for X

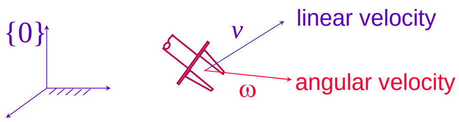
Given a representation $x = \begin{bmatrix} x_P \\ x_R \end{bmatrix}$

$$\dot{x} = J_x(q)\ \dot{q}$$

$$J_x(q) = E(x)\ J_0(q)$$

Basic Jacobian $\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q)\ \ \dot{q}$

Basic Jacobian



$$\begin{pmatrix} v \\ \omega \end{pmatrix}_{(6x1)} = J_0(q)_{(6xn)}\,\dot{q}_{(nx1)}$$

Jacobian and Basic Jacobian

$$J = \begin{pmatrix} J_{XP} \\ J_{XR} \end{pmatrix} = \begin{pmatrix} E_P & 0 \\ 0 & E_R \end{pmatrix} \begin{pmatrix} J_v \\ J_w \end{pmatrix}$$

$$J(q) = E(X) \quad J_0(q)$$

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = J_0(q)\ \dot{q}$$

Position Representations

$$\dot{x}_P = E_P(x_P)\ v$$

Cartesian Coordinates (x, y, z)

$$E_P(x) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

Using $(x \ \ y \ \ z)^T = (\rho\cos\theta \ \ \rho\sin\theta \ \ z)^T$

$$E_P(X) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta/\rho & \cos\theta/\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

Using

$$(x \ \ y \ \ z)^T = (\rho\cos\theta\sin\phi \ \ \rho\sin\theta\sin\phi \ \ \rho\cos\theta)^T$$

$$E_P(X) = \begin{pmatrix} \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \\ -\sin\theta/(\rho\sin\phi) & \cos\theta/(\rho\sin\phi) & 0 \\ \cos\theta\cos\phi/\rho & \sin\theta\cos\phi/\rho & -\sin\phi/\rho \end{pmatrix}$$

Position Representations (inverse)

$$v = E_P^{-1}(x)\dot{x}_p$$

Cartesian Coordinates (x, y, z)

$$E_P^{-1}(X) = I_3$$

Cylindrical Coordinates (ρ, θ, z)

$$E_P^{-1}(X) = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ -\sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Spherical Coordinates (ρ, θ, ϕ)

$$E_P^{-1}(X) = \begin{pmatrix} \cos\theta\sin\phi & \rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ -\sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$$

Rotation Representations

Direction Cosines

$$x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} ; \; E_r(x_r) = \begin{pmatrix} -\hat{r}_1 \\ -\hat{r}_2 \\ -\hat{r}_3 \end{pmatrix}$$

$$\dot{x}_r = E_r \omega$$

Direction Cosines - Rotation Error

Instantaneous Angular Error

$$x_r = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} ; \; x_{rd} = \begin{bmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{bmatrix}$$

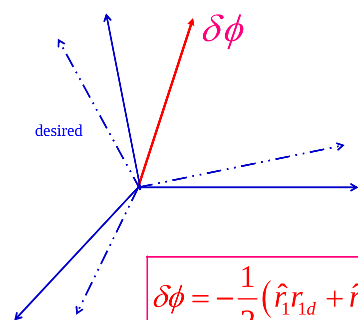
$$\delta x_r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} - \begin{pmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{pmatrix}$$

$$\omega = \frac{1}{2} E^T \dot{x}_r$$

$$\delta\phi = \frac{1}{2} E^T \delta x_r$$

$$\delta x_r = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} - \begin{pmatrix} r_{1d} \\ r_{2d} \\ r_{3d} \end{pmatrix}$$

Instantaneous Angular Error



$$\delta\phi = -\frac{1}{2}\left(\hat{r}_1 r_{1d} + \hat{r}_2 r_{2d} + \hat{r}_3 r_{3d}\right)$$

Euler Angles

$$E_r(X) = \begin{pmatrix} -{}^{S\varphi C\theta}/_{S\theta} & {}^{C\varphi C\theta}/_{S\theta} & 1 \\ C\varphi & S\varphi & 0 \\ {}^{S\varphi}/_{S\theta} & -{}^{C\varphi}/_{S\theta} & 0 \end{pmatrix}$$

$$E_r^{-1}(x_r) = \begin{pmatrix} 0 & \cos\psi & \sin\psi \sin\theta \\ 0 & \sin\psi & -\cos\psi \sin\theta \\ 1 & 0 & \cos\theta \end{pmatrix}$$

Euler Parameters

$$x_r = \lambda = \left(\lambda_0 \; \lambda_1 \; \lambda_2 \; \lambda_3\right)^T$$

$$\dot{\lambda} = E\omega$$

$$E = \frac{1}{2} \begin{pmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & \lambda_3 & -\lambda_2 \\ -\lambda_3 & \lambda_0 & \lambda_1 \\ \lambda_2 & -\lambda_1 & \lambda_0 \end{pmatrix}$$

Euler Parameters

$$E_r^+(x_r) = 2\begin{pmatrix} -\lambda_1 & \lambda_0 & -\lambda_3 & \lambda_2 \\ -\lambda_2 & \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_3 & -\lambda_2 & \lambda_1 & \lambda_0 \end{pmatrix}$$

$$\boxed{\delta\phi = E^+ \lambda_d}$$

Joint Space Dynamics

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \Gamma$$

- q: Generalized Joint Coordinates
- $M(q)$: Mass Matrix - Kinetic Energy Matrix
- $V(q,\dot{q})$: Centrifugal and Coriolis forces
- $G(q)$: Gravity forces
- Γ: Generalized forces

PD Control Stability

$$M(q)\ddot{q} + B(q)[\dot{q}\dot{q}] + C(q)[\dot{q}^2] + G(\theta) = \tau$$

$$\tau = -k_p(q - q_d) - k_v\dot{q}$$

$$V_d = 1/2k_p(q-q_d)^2$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} + \frac{\partial V_s}{\partial q} = \not{\tau}\, \frac{\partial V_d}{\partial q} - k_v\dot{q}$$

PD Control Stability

$$M(q)\ddot{q} + B(q)[\dot{q}\dot{q}] + C(q)[\dot{q}^2] + G(\theta) = \tau$$

$$\tau = -k_p(q - q_d) - k_v\dot{q}$$

$$V_d = 1/2k_p(q-q_d)^T(q-q_d)$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} + \frac{\partial (V_s - V_d)}{\partial q} = \tau_s$$

$\tau_s = -k_v\dot{q}$ with $\boxed{\tau_s^T\dot{q} < 0 \;\; for\; \dot{q} \neq 0;\; k_v > 0}$

Performance

High Gains $\longrightarrow$ better disturbance rejection

Gains are limited by

- structural flexibilities
- time delays (actuator-sensing)
- sampling rate

$$\omega_n \le \frac{\omega_{res}}{2} \quad \longleftarrow \text{lowest structural flexibility}$$

$$\omega_n \le \frac{\omega_{delay}}{3} \quad \longleftarrow \text{largest delay} \left(\frac{2\pi}{\tau_{delay}}\right)$$

$$\omega_n \le \frac{\omega_{sampling-rate}}{5}$$

Nonlinear Dynamic Decoupling

$$M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) = \tau$$

$$\tau = \hat{M}(\theta)\underline{\tau'} + \hat{V}(\theta,\dot{\theta}) + \hat{G}(\theta)$$

$$\mathbf{1}.\ddot{\theta} = (M^{-1}\hat{M})\tau' + M^{-1}[(V-\hat{V}) + (G-\hat{G})]$$

with perfect estimates

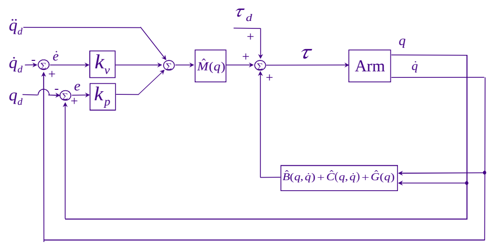
$$\mathbf{1}.\ddot{\theta} = \tau' + \varepsilon(t)$$

τ': input of the unit-mass systems

$$\tau' = \ddot{\theta}_d - k_v'(\dot{\theta} - \dot{\theta}_d) - k_p'(\theta - \theta_d)$$

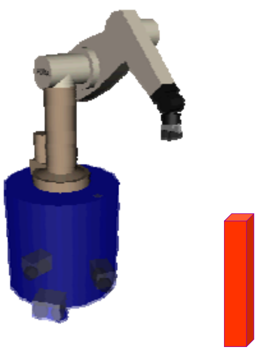
Closed-loop

$$\ddot{E} + k_v'\dot{E} + k_p'E = 0 + \varepsilon(t)$$



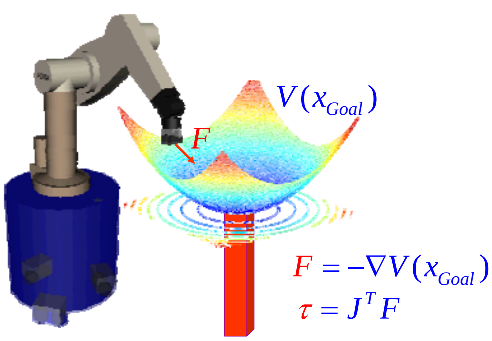
Task Oriented Control

Joint Space Control

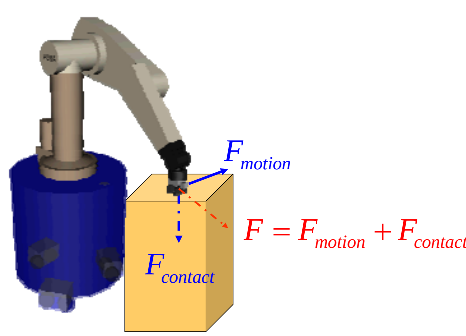


Joint Space Control

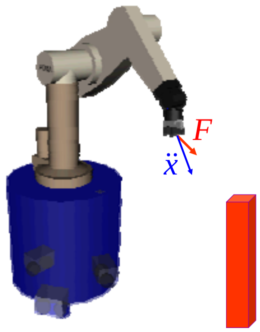
Operational Space Control



Unified Motion & Force Control



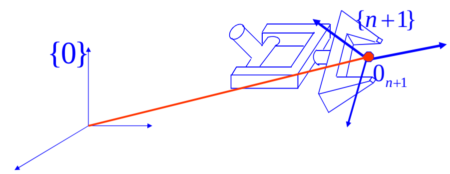
Operational Space Dynamics



$$M_x \ddot{x} + V_x + G_x = F$$

$$F = F[\hat{M}_x, \hat{V}_x, \hat{G}_x, V(x_{Goal})]$$

Task-Oriented Equations of Motion



Non-Redundant Manipulator ; $n = m$

$$x = (x_1 \, x_2 \ldots x_m)^T$$

$$q = (q_1 \, q_2 \ldots q_n)^T$$

Operational Space Dynamics

$$M_x(x)\ddot{x} + V_x(x, \dot{x}) + G_x(x) = F$$

x:	End-Effector Position and Orientation
$M_x(x)$:	End-Effector Kinetic Energy Matrix
$V_x(x, \dot{x})$:	End-Effector Centrifugal and Coriolis forces
$G_x(x)$:	End-Effector Gravity forces
F:	End-Effector Generalized forces

Joint Space/Task Space Relationships

Kinetic Energy

$$K_x(x, \dot{x}) \equiv K_q(q, \dot{q})$$

$$\frac{1}{2}\dot{x}^T M_x(x)\dot{x} \equiv \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

Using $\dot{x} = J(q)\dot{q}$

$$\frac{1}{2}\dot{q}^T (J^T M_x J)\, \dot{q} \equiv \frac{1}{2}\dot{q}^T M \, \dot{q}$$

Joint Space/Task Space Relationships

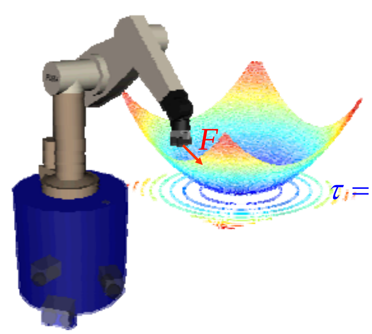
$$M_x(x) = J^{-T}(q) M(q) J^{-1}(q)$$

$$V_x(x, \dot{x}) = J^{-T}(q) V(q, \dot{q}) - M_x(q) h(q, \dot{q})$$

$$G_x(x) = J^{-T}(q) G(q)$$

where $h(q, \dot{q}) \doteq \dot{J}(q)\dot{q}$

End-Effector Control



$$\tau = J^T(q) F$$

Passive Systems (Stability)

$$V_{goal} = \frac{1}{2} k_p \left(x - x_g\right)^T \left(x - x_g\right)$$

System $\quad \frac{d}{dt}\left(\frac{\partial (K-V)}{\partial \dot{x}}\right) - \frac{\partial (K-V)}{\partial x} = F$

$$\Downarrow \; F = -\frac{\partial}{\partial X}\left(V_{goal} - \hat{V}\right)$$

Conservative Forces

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) - \frac{\partial \left(K - V_{goal}\right)}{\partial x} = 0$$

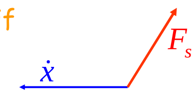
Stable

Asymptotic Stability

a system $\quad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) - \frac{\partial \left(K - V_{goal}\right)}{\partial x} = 0 \; F_s$

is asymptotically stable if

$$F_s^T \dot{x} < 0 \quad ; \; for \; \dot{x} \neq 0$$



$$F_s = -k_v \dot{x} \;\rightarrow\; k_v > 0$$

Control

$$F = -k_p \left(x - x_{goal}\right) + \hat{G}_x - k_v \dot{x}$$

Nonlinear Dynamic Decoupling

Model

$$M_x(x)\ddot{x} + V_x(x,\dot{x}) + G_x(x) = F$$

Control Structure

$$F = \hat{M}(x) F' + \hat{V}_x(x,\dot{x}) + \hat{G}_x(x)$$

Decoupled System

$$I \ddot{x} = F'$$

with $\quad \tau = J^T F$

Perfect Estimates

$$I \ddot{x} = F'$$

F' input of decoupled end-effector

Goal Position Control

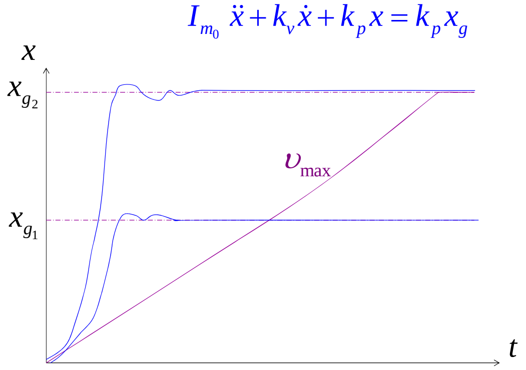
$$F' = -k_v' \dot{x} - k_p' \left(x - x_g\right)$$

Closed Loop

$$I \ddot{x} + k_v' \dot{x} + k_p' (x - x_g) = 0$$

Closed Loop

$$I_{m_0} \ddot{x} + k_v \dot{x} + k_p x = k_p x_g$$

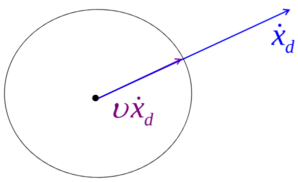


PD Control

$$F^* = -k_v \dot{x} - k_p \left(x - x_g\right)$$

Velocity-Like Control

$$F^* = -k_v \left(\dot{x} - \frac{k_p}{k_v}\left(x_g - x\right)\right)$$



$$\dot{x}_d = \frac{k_p}{k_v}\left(x_g - x\right)$$

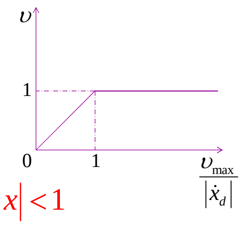
$$F^* = -k_v\Big(\dot{x} - \underbrace{\frac{k_p}{k_v}\left(x_g - x\right)}_{\dot{x}_d}\Big)$$

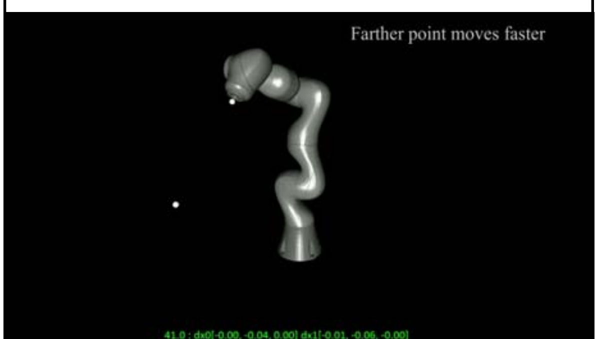
$$F^* = -k_v\left(\dot{x} - \upsilon \dot{x}_d\right)$$

with

$$\upsilon = sat\left(\frac{V_{\max}}{\left|\dot{x}_d\right|}\right)$$

$$sat(x) = \begin{cases} x & if\ |x| < 1 \\ sign(x) & if\ |x| > 1 \end{cases}$$





Trajectory Tracking

Trajectory: x_d , $\dot{x}_d$, $\ddot{x}_d$

$$F' = I\ \ddot{x}_d - k_v'(\dot{x} - \dot{x}_d) - k_p'(x - x_d)$$

$$(\ddot{x} - \ddot{x}_d) + k_v'(\dot{x} - \dot{x}_d) + k_p'(x - x_d) = 0$$

or $\ddot{\varepsilon}_x + k_v'\dot{\varepsilon}_x + k_p'\varepsilon_x = 0$

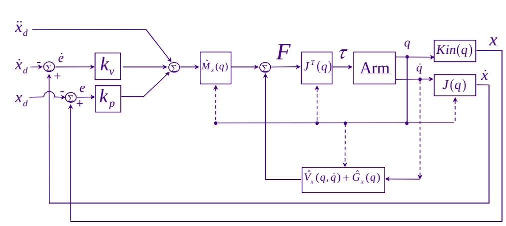
with $\varepsilon_x = x - x_d$

In joint space

$$\ddot{\varepsilon}_q + k_v'\dot{\varepsilon}_q + k_p'\varepsilon_q = 0$$

with $\varepsilon_q = q - q_d$

Task-Oriented Control

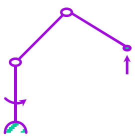


Compliance

$$I\ \ddot{x} = F'$$

$$F' = -\begin{pmatrix} k'_{p_x} & 0 & 0 \\ 0 & k'_{p_y} & 0 \\ 0 & 0 & k'_{p_z} \end{pmatrix}(x - x_d) - k'_v \dot{x}$$

set to zero



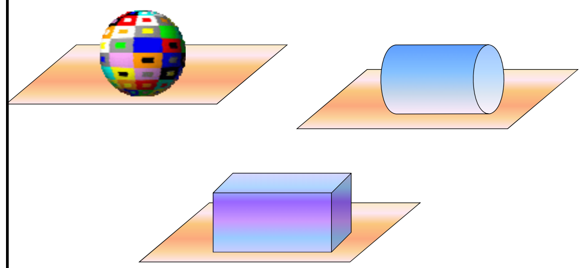
$$\ddot{x} + k'_v \dot{x} + k'_{px}(x - x_d) = 0$$

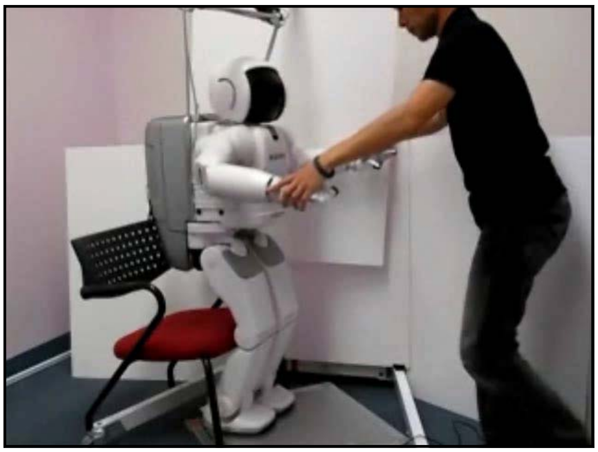
$$\ddot{y} + k'_v \dot{y} + k'_{py}(y - y_d) = 0$$

$$\ddot{z} + k'_v \dot{z} = 0$$

Compliance along Z

Task Description

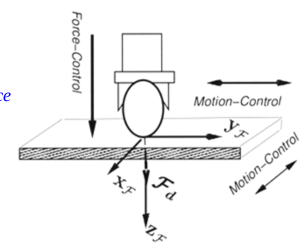




Task Specification

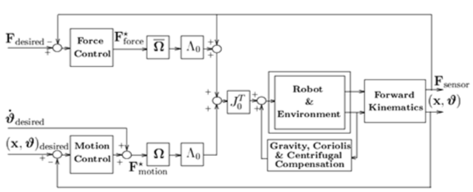
$$F = \Omega F_{motion} + \bar{\Omega} F_{force}$$

Selection matrix



$$\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \bar{\Omega} = I - \Omega$$

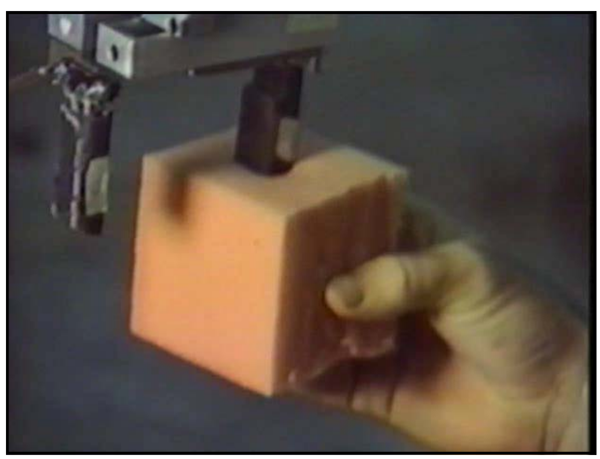
Unified Motion & Force Control



Two decoupled Subsystems

$$\Omega \dot{\vartheta} = \Omega F^*_{motion}$$

$$\bar{\Omega} \dot{\vartheta} = \bar{\Omega} F^*_{force}$$



System Identification

Natural Systems

Conservative Systems

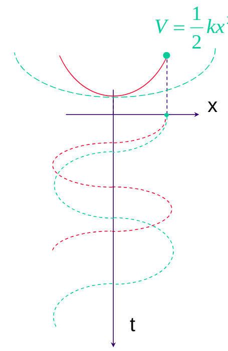
$$m\ddot{x} + kx = 0$$

Frequency increases
with stiffness
and inverse mass

Natural Frequency $\omega_n = \sqrt{\frac{k}{m}}$

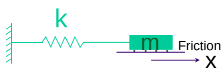
$$\ddot{x} + \omega_n^2 x = 0$$

$$x(t) = c\ \cos(\omega_n t + \phi)$$



Natural Systems

Dissipative Systems

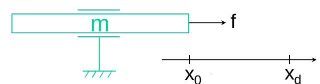


$$\frac{d}{dt}\left(\frac{\partial(K-V)}{\partial \dot{x}}\right) - \frac{\partial(K-V)}{\partial x} = f_{friction}$$

Viscous friction: $f_{friction} = -b\dot{x}$

$$m\ddot{x} + b\dot{x} + kx = 0$$

Identification



System

$$m\ddot{x} + b\dot{x} = f$$

Control

$$f = -k_p(x - x_d) - k_v\dot{x}$$

Closed-Loop

$$m\ddot{x} + (b + k_v)\dot{x} + k_p(x - x_d) = 0$$

Time Response

$$\ddot{x} + 2\xi_n\omega_n\dot{x} + \omega_n^2 x = 0$$

$$\omega_n = \sqrt{\frac{k_p}{m}} \Rightarrow m = \frac{k_p}{\omega_n^2} \qquad \xi_n = \frac{b + k_v}{2\sqrt{k_p m}}$$

$$x(t) = ce^{-\xi_n\omega_n t}\cos(\omega_n\sqrt{1-\xi_n^2}\,t + \phi)$$