1. Consider the 1-DOF system described by the equation of motion, $4\ddot{x} + 20\dot{x} + 25x = f$.

(a) Find the natural frequency $\omega_n$ and the natural damping ratio $\zeta_n$ of the natural (passive) system ($f = 0$). What type of system is this (oscillatory, over-damped, etc.)?

(b) Design a PD controller that achieves critical damping with a closed-loop stiffness $k_{CL} = 36$. In other words, let $f = -k_v\dot{x} - k_px$, and determine the gains $k_v$ and $k_p$. Assume that the desired position is $x_d = 0$. 
(c) Assume that the friction model changes from linear \(20\dot{x}\) to Coulomb friction, \(30\text{sign}(\dot{x})\). Design a control system which uses a non-linear model-based portion with trajectory following to critically damp the system at all times and maintain a closed-loop stiffness of \(k_{CL} = 36\). In other words, let \(f = \alpha f' + \beta\) and \(f' = \ddot{x}_d - k'_v(\dot{x} - \dot{x}_d) - k'_p(x - x_d)\). Then, find \(f, \alpha, \beta, f', k'_v\) and \(k'_p\). Note that \(f\) is an \(m\)-mass control, and \(f'\) is a unit-mass control. Use the definition of error, \(e = x - x_d\).

(d) Given a disturbance force \(f_{dist} = 4\), what is the steady-state \((\ddot{e} = \dot{e} = 0)\) error of the system in part (c)?
2. For a certain RR manipulator, the equations of motion are given by

\[
\begin{bmatrix}
4 + c_2 & 1 + c_2 \\
1 + c_2 & 1
\end{bmatrix}
\begin{bmatrix}
\ddot{\theta}_1 \\
\ddot{\theta}_2
\end{bmatrix}
+ \begin{bmatrix}
-s_2(\dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2) \\
2s_2\dot{\theta}_1^2
\end{bmatrix}
= \begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix}
\]

(a) Assume that joint 2 is locked at some value \( \theta_2 \) using brakes and joint 1 is controlled with a PD controller, \( \tau_1 = -40\dot{\theta}_1 - 400(\theta_1 - \theta_{1d}) \). What is the minimum and maximum inertia perceived at joint 1 as we vary \( \theta_2 \)? What are the corresponding closed-loop frequencies?

(b) Still assuming that joint 2 is locked, at what values of \( \theta_2 \) do the minimum and maximum damping ratios occur?
(c) Now assume that both joints are free to move, and that this system is controlled by a partitioned PD controller, \( \tau = \alpha \tau' + \beta \). Design a partitioned, trajectory-following controller (one that tracks a desired position, velocity and acceleration) which will provide a closed-loop frequency of 10 rad/sec on joint 1 and 20 rad/sec on joint 2 and be critically damped over the entire workspace. That is, let
\[
\tau' = \ddot{\theta}_d - \begin{bmatrix} k'_{p_1} & 0 \\ 0 & k'_{p_2} \end{bmatrix} \left( \dot{\theta} - \dot{\theta}_d \right) - \begin{bmatrix} k'_{v_1} & 0 \\ 0 & k'_{v_2} \end{bmatrix} \left( \theta - \theta_d \right)
\]
then find the matrices \( \alpha \) and \( \beta \) and the vector \( \tau \), along with the necessary gains \( k'_{v_i} \) and \( k'_{p_i} \).

(d) If \( \theta_2 = 180^\circ \), what is the steady-state error vector for a given disturbance torque, \( \tau_{dist} = [2 \ 4]^T \)?