1. In the derivation of the kinetic energy matrix, we found that the kinetic energy of a link \( i \) is given by

\[
K_i = \frac{1}{2} \left( m_i v_{C_i}^T v_{C_i} + \omega_i^T C_i \omega_i \right)
\]

where \( m_i \) is the mass of the link, \( v_{C_i} \) is the velocity of the center of mass of the link, \( \omega_i \) is the angular velocity of the link, and \( C_i \) is the inertia tensor of the link expressed in some frame \( \{C\} \) attached to the center of mass. This equation doesn’t mention any specific frames, however. In this problem, we will show that the frames that we use to calculate the mass matrix don’t matter, as long as we’re consistent.

(a) Let there be two frames \( \{A\} \) and \( \{B\} \). \( A v_{C_i} \) is the velocity with respect to a ground frame of the center of mass of link \( i \) expressed in basis vectors of frame \( \{A\} \). Show that the contribution of the linear motion to the kinetic energy of this link expressed in frame \( \{B\} \) is equal to that expressed in frame \( \{A\} \).
(b) Assume that we have a frame \( \{ C_i \} \), located at the center of mass of link \( i \) and that we have \( C_i I_i \) the inertia tensor of the link expressed in \( \{ C_i \} \). \( C_i \omega_i \) is the angular velocity of the link with respect to a ground frame expressed in basis vectors of frame \( \{ C_i \} \). Let there also be another frame \( \{ C'_i \} \), which has the same origin as frame \( \{ C_i \} \). Show that the contribution of the angular motion to the kinetic energy of this link expressed in frame \( \{ C'_i \} \) is equal to that expressed in frame \( \{ C_i \} \).

(c) What generalizations can be made about the kinetic energy computation based on the results derived in parts (a) and (b).
2. Equations of Motion - Modeling the dynamics of a certain RRP manipulator, we arrive at the following equations of motion:

\[
\begin{align*}
\tau_1 &= (m_1l^2/4 + m_2l^2 + m_3(l^2 + d_3^2s_2^2) + I_{ZZ_1})\ddot{\theta}_1 + m_3ld_3c_2\ddot{\theta}_2 + m_3ls_2d_3 + 2m_3d_3^2s_2c_2\dot{\theta}_1\dot{\theta}_2 \\
&\quad + 2m_3d_3s_2^2\ddot{\theta}_1d_3 + 2m_3lc_2\ddot{\theta}_2d_3 - m_3ld_3s_2^2\dot{\theta}_2 + b_1\dot{\theta}_1 \\
\tau_2 &= ??? + m_3d_3^2\ddot{\theta}_2 + 0\ddot{d}_3 + 2m_3d_3\dot{\theta}_2d_3 - m_3d_3^2s_2c_2\dot{\theta}_1^2 - m_3d_3s_2g + b_2\dot{\theta}_2 \\
\tau_3 &= m_3ls_2\ddot{\theta}_1 + 0\ddot{\theta}_2 + m_3(c_2 + 1)d_3 + 2m_3d_3\dot{\theta}_2^2d_3 - m_3d_3s_2^2\dot{\theta}_1^2 - m_3d_3\dot{\theta}_2^2 + m_3c_2g + b_3\ddot{d}_3.
\end{align*}
\]

A term, indicated by “???” , has been left out.

(a) Give a Coriolis term acting at joint 2.

(b) Give a centrifugal term acting at joint 2.

(c) Give a gravity term acting at joint 2.

(d) Give a friction term acting at joint 2.

(e) Find the missing term “???” and explain how you found it. (Hint: write the mass matrix, \(M(q)\).) 

(f) Find the two incorrect terms acting at joint 3 and explain how you found them.
3. You are given the following PR manipulator. Gravity acts in the $-Z_0$ direction. The links are made of solid, uniform square bars with sides of length $a$ and density $\rho$. Assume the joints themselves are massless.

You are GIVEN that the mass matrix for the manipulator is:

$$M = \begin{bmatrix} m_1 + m_2 & -\frac{1}{2}L_2\sin(\theta_2)m_2 \\ -\frac{1}{2}L_2\sin(\theta_2)m_2 & I_{zz2} + \frac{1}{4}L_2^2m_2 \end{bmatrix}$$

(a) Calculate the C and B matrices for this manipulator, representing the Coriolis and centrifugal components of the dynamic forces of the manipulator.

(b) Write down the gravity vector in frame $\{0\}$, $^0G(q)$. (Hint: For this simple manipulator, it is not necessary to compute any Jacobians to do this).
(c) Calculate the inertia tensor of each link about its own center of mass, that is, find $C_1 I_1$ and $C_2 I_2$. You may use the formula from a lookup table, but if you have never done the integration yourself, it is worth the time to do it.

(d) Explain why only $I_{zz2}$ showed up in the mass matrix given above. What about the other non-zero terms in $C_1 I_1$ and $C_2 I_2$?

(e) Write the equations of motion for this manipulator, substituting your expressions for $m_1$, $m_2$, and $I_{zz2}$. (e.g., if we wanted a computer program to output joint torques at a particular time step, we would write $\tau_1 = \ldots$, and $\tau_2 = \ldots$.)